MBD

Relativity & Cosmology

For BS & M.Sc (Physics)

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COURSE OUTLINES

(Govt. College University, Faisalabad)

PHY-509 Relativity and Cosmology 3(3–0)

Special Relativity: Galilean relativity, Einstein's postulates of special relativity. Consequences of special relativity. Michelson-Morley experiment, Lorentz transformations, consequences of Lorentz transformations (the relativity of length, the relativity of time, concept of simultaneity).

Relativistic Mechanics: Transformation of relativistic velocities, addition of relativistic velocities, Relativity of mass, Force equation in relativity, rest mass, KE and total energy, conservation of energy and momentum, the conversion of mass to kinetic energy in Uranium Fission, pair production and annihilation, The Cerenkov Effect and Cerenkov radiation, Einstein's mass-energy relationship and its practical examples, particle of Zero rest mass, Relativistic Doppler Effect, Aberration of Light, Tachyons.

Elementary Tensor: Structure of space time, Geometry of space time, Minkowski spacetime, the light cone, tensors and four vectors (position four vector, velocity four vector, momentum four vector, force four vector). Manifolds and coordinates, curves and surfaces, tensor fields, metric tensor.

General Relativity: Gravity as a Geometry, The equivalence principle, clocks in a gravitation field, spacetime is curved, geodesics, the geodesic equation, equation of geodesic deviation, Einstein field equation.

Cosmology: Cosmological redshift, Hubble's law, The Big Bang Theory, Historical background of universe, Stars, Neutron stars, Pulsars, Black holes, Singularity, Quasars, Microwave background, Measuring the distance to stars, Concept of open, closed and flat universe, Dark matter (MACHOs and WIMPs).

Recommended Books:

1. Goldstein H, P Charle and J L Safko. 2001, Classical Mechanics. Addison-Wesley.
2. Hand L N and J.D Finch. 1998, Analytical Mechanics. Cambridge University Press, Cambridge.
3. Barger V D and M G Olsson. 1995, Classical Mechanics. McGraw-Hill, New York.
4. Landau L D and E M Lifshitz. Mechanics. 1960, Pergamon, Oxford.
5. Classical Mechanics by J.W. Leech Methuen and Co. Ltd., London, 1958.

(University of Sargodha, Sargodha)

PHYS-608 Relativity and Cosmology 3 (3 – 0)

Special Relativity: Galilean Transformations, Existence of Ether, Michelson-Morley Experiment, Stellar Aberration, Einstein's Postulates of Special Relativity, Lorentz Transformations, Relativity of Simultaneity, Time Dilation (Twins Paradox), Length Contraction (Ladder Paradox), Velocity Transformation and Velocity Addition, Relativistic Mechanics, Minkowski Spacetime, Line Element, Four-Vectors, Force Equation in relativity, Rest Mass, Kinetic and Total Energy, Conservation of Energy and Momentum.

Tensor: Elements of Tensor Calculus, Manifolds and Coordinates, Curves and Surfaces, Tensor Fields, Lie Derivative, Geodesics, Parallel Transport, Riemann Tensor, Metric Tensor, Christoffel Symbols.

General Relativity: Principles of General Relativity, Equation of Geodesics, Einstein's Field Equations.

Cosmology: Newtonian Cosmology, Cosmological Redshift, Hubble's Law, The Big Bang, Expansion Rate.

Recommended Books:

1. Dynamics and Relativity, by W.D.McComb, Oxford University Press, (1999).
2. Introduction to Cosmology, by J.V.Narlikar, Cambridge University Press, (1989).
3. Introducing Einstein's Relativity, by R.D.D'Inverno, Oxford University Press, (1992).
4. The Special Theory of Relativity by D. Bohm. Routledge (2009).
5. General Theory of Relativity by P. M. Dirac. John Wiley and Sons (1975).
6. Theory of Relativity based on Physical Reality by L. Janossy Academia Kiado, Budapest (1971).
7. Relativity Demystified D. MacMohan Mc Graw-Hill (2006).
8. Dynamics and Relativity by J. R. Forshaw and A. G. Smith. John Wiley and Sons (2009).

(University of Punjab, Lahore)

PHY-404 Relativity and Cosmology 3 (3 – 0)

Special Relativity: Galilean relativity, concept of ether, Michelson-Morley experiment, Einstein's postuiates of special relativity, Lorentz transformations, structure of spacetime, Minkowski spacetime tensors, the light-cone, line element, four-vectors, relativity of simultaneity, time dilation, proper time, length contraction, twin paradox, velocity transformation and velocity addition. Relativistic Mechanics, Force equation in relativity, rest mass, kinetic and total energy, conservation of energy and momentum.

Elements of Tensor Calculus: Manifolds and coordinates, curves and surfaces, tensor fields, geodesics, Riemann tensor, metric tensor, Einstein's tensor.

General Relativity: Principles of general relativity, equation of geodesics deviation, Einstein's field equations.

Cosmology: Newtonian cosmology, cosmological redshift, Hubble's law, microwave background, the Big Bang, FRW metric.

Recommended Books:

1. Dynamics and Relativity, by W. D. McComb, OxfordUniversity Press, 1999.
2. Introduction to Cosmology, J. V. Narlikar, CambridgeUniversity Press, 1989.
3. Introducing Einstein's Relativity, R. D'Inverno, Oxford University Press, 1992.

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Part I

Special Relativity

Chapter 1

Special Theory of Relativity

Learning Objectives

In this chapter you will learn

- Basic terms of special relativity including frames of reference, observers and events.
- How to transform equations from one frame of reference to other frames of reference using Galilean transformations.
- How the concept of aether was rejected by Michelson-Morley experiment and failure of Galilean transformations to explain the constancy of velocity of light.
- Two postulates of special theory of relativity.
- How Lorentz transformations explained the constancy of velocity of light and replaced the Galilean transformations.
- How time dilation explains the twin paradox.
- How length contraction explains the ladder paradox.
- Why do observers in relative motion disagree sometimes about the simultaneity of two events.

KEY TERMS

Frame of reference, Observer, Event, Galilean transformations, Relative velocity, Aether, Gamma factor, Lorentz boost, Time dilation, Twin paradox, Length contraction, Ladder paradox

1.1 Introduction

Q. Give a brief introduction of special theory of relativity.

The physical theory regarding the relative motion and relationship between space and time is called special relativity.

During European Renaissance, *Tycho Brahe* made an accurate measurements of motion of planets in our solar system.

Kepler, between 1609-19, analysed this data and proposed three laws of planetary motion.

Galileo invented the telescope and gave advance observations in Astrophysics and Cosmology. He formulated the 'principle of relativity'[1] in order to show that one cannot determine whether the earth revolves about the sun or the sun revolves about the earth. He believed that no experiment in mechanics could reveal whether you are in uniform motion or at rest; all mechanics experiments work the same way regardless of your motion.

Newton, in 1687, developed three laws of motion in Principia on the basis of Galileo's principle of relativity. This principle was rephrased by Pioncaré later as "The impossibility of demonstrating absolute motion in a general law of nature".

Maxwell, in 1873, formulated a unified theory of electricity and magnetism. This theory explain that light is the propagation of an electromagnetic disturbance through space with a velocity of $3 \times 10^8 ms^{-1}$. In this way electromagnetic theory is unified with optics.

Michelson and Morley, in 1887, suspected that velocity of light in a vacuum would be invariant irrespective of motion of measuring equipment.

Einstein[2], in 1905, solved the problem of invariance of light in his paper 'on the electrodynamics of moving bodies' which was later named as 'special theory of relativity'. This was the first theory which was consistent with Maxwell's electromagnetic unified theory. This theory only deals with inertial frames of reference. It generalizes Galileo's principle of relativity. He claimed in this theory that space and time are not independent coordinates but must be treated on equal footing. It is not applicable to accelerating frames (non-inertial) or gravitational field. The effect of gravity was later included by Einstein in 1916 in general theory of relativity.

Special theory of relativity has impact on many areas of physics including thermodynamics, electromagnetism, optics, atomic physics, nuclear physics and high-energy physics.

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[1] **❝The laws of mechanics are same in all frames of reference in uniform motion❞.**

[2] **Albert Einstein (1879-1955) developed special theory of relativity (1905) and general theory of relativity (1916). He received the 1921 Nobel prize in physics.**

1.2 Basic Terminologies of Special Relativity

Q. Define some basic concepts of relativity; frames of reference, observers and event.

1.2.1 Frames of Reference

> A laboratory or place where some phenomena are observed and measurements are taken is called frame of reference.

- The frame of reference is the place or the object attached with us. In everyday life ground and vehicles are the frames of reference. Frame of reference is important to perform an experiment when experiments observe what is going on in their own frame of reference. The same experiments give identical observations. When they look into other's frame of reference, even though both the frames of reference are in uniform motion. i.e., motion with constant velocities, they observe differently. This fact introduces new concepts and becomes the back-bone of the theory of relativity. There are two types of frame of reference:

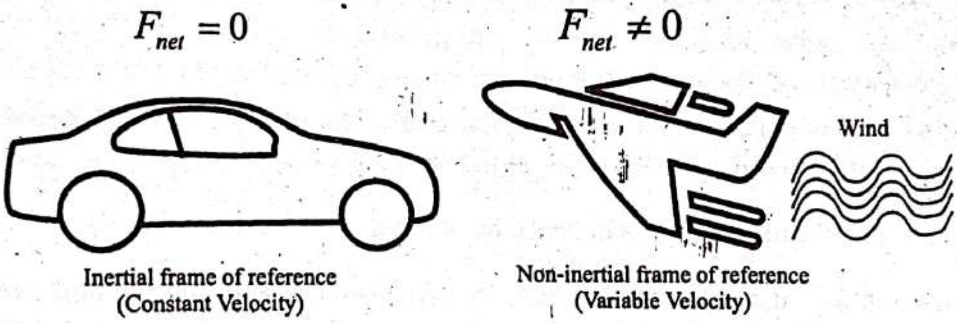

Figure 1.1: *Inertial and non-inertial frame of reference*

1. Inertial frame of reference
2. Non-inertial frame of reference

Inertial Frame of Reference (a = 0)

> The coordinate system which is moving with uniform velocity or at rest is called inertial frame of reference.

If a body is subject to no external force then frame of reference will be inertial frame of reference. It is non-accelerating frame of reference and valid for flat surfaces. Usually, flat spacetime structure is treated as inertial. If Newton's laws are valid in one inertial reference frame then they are valid too in any inertial frame. Examples of inertial frame of reference are a car moving with uniform velocity, Earth (ground)[3].

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[3] Earth is actually non-inertial frame if seen globally.

3. The origins of S and S' coincide at $t = 0$ and $t' = 0$ in S and S' frames respectively.

In general, the observers in S and S' frames of reference will not agree about coordinates of an event but there must exist a set of equations relating the coordinates (x, y, z, t) for S frame of reference and the coordinates (x', y', z', t') for S' frame of reference.

1.3.1 The Galilean Transformations

The coordinate transformations, in classical physics, which connects one inertial frame of reference with another inertial frame of reference are called Galilean transformations.

Before Einstein's special theory of relativity, coordinates of frames of reference in relative motion were related by the Galilean transformation equations. Let us consider an event (Fig.2.1) is taking place in frame of reference S at coordinates (x, y, z, t) moving with velocity[6] v along +ve x-axis will be observed in frame of reference S' at coordinates (x', y', z', t'). The coordinates of both frames of reference will be related by Galilean transformation equations given as

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}$$

Here time is absolute. At any time t the origin of S' will travel a distance vt along the x-axis as measured in frame of reference S. Any event that occurs at time t with position coordinate x in frame S must occur at $x' = x - vt$ in frame S', while the values of x, y and $z-$axes will remain unaffected. These equations are written by assuming that observers in S and S' frames observe the event at same time. i.e., $t = t'$ (time is absolute).

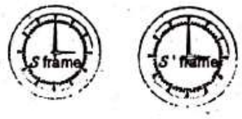

Figure 1.3: *Time in rest (left) and moving (right) frames is same*

The equation $(t = t')$ infers that time passes at the same rate in both frames of reference (Fig.1.3). But Einstein claimed that time passes at different rate in both frames of reference. He explained that time is relative rather absolute. This is only assumption of absolute time. Ultimately Galilean transformation equations became limited.

Relative Velocity

The velocity of an object in a frame of reference relative to other frame of reference is called relative velocity. For the motion of a car, in different frames of reference as described in above topic, the velocity

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[6] In this book v will be used for velocity of frame of reference and u as velocity of object in relative motion.

$u(u_x, u_y, u_z)$ in a rest frame of reference S can be related with the velocity $u'(u'_x, u'_y, u'_z)$ in another frame of reference moving along $x-$axis with relative velocity v as

$$\begin{aligned}
u'_x &= \frac{dx'}{dt'} = \frac{d(x-vt)}{dt} \equiv u_x - v \\
u'_y &= \frac{dy'}{dt'} = \frac{dy}{dt} \equiv u_y \\
u'_z &= \frac{dz'}{dt'} = \frac{dz}{dt} \equiv u_z
\end{aligned}$$

Like spatial coordinates, velocity component along $x-$axis in frame of reference S is different from that in frame of reference S'. But velocity component along y and $z-$axes remain unaffected. This is because there is no motion along y and $z-$axes. If there is a motion along y and $z-$axes too then velocity components along y and $z-$axes would change too.

Maxwell's equations implies the velocity of light is $3 \times 10^8 ms^{-1}$ but according to Galilean relativity velocity of light would be $(c + v)$ or $(c - v)$ in a frame moving with velocity v. Here Galilean relativity is not consistent with Maxwell's equations in electromagnetism. So Galilean relativity is applicable to only mechanics.

1.4 Aether

Q. What is ether theory? Also describe Michelson-Morley experiment in details.

> The hypothetical substance through which electromagnetic waves travel is called aether.

Aether was proposed by the Greek philosopher Aristotle and was used in several optical theories. It was assumed to be massless and transparent for light and other electromagnetic waves. It was only a way to allow propagation of light and propagation was impossible in empty space.

In 1887, *Michelson* and *Morley* performed an experiment to detect the existence of the aether. The experiment shocked the scientific community by yielding results which implied the non-existence of aether. The experiment concluded that velocity of light is same in rest and moving frames of reference in aether. The experiment concluded that velocity of light is same in rest and moving frames of reference in contradiction with Galilean velocity transformation equations. This result was later on used by Einstein to refuse the existence of the aether and allowed him to develop special theory of relativity.

1.4.1 Michelson-Morley Experiment

Michelson and Morley, in 1887, performed an experiment to detect the motion of the Earth relative to the aether. It was an attempt to discover the absolute frame of reference. They found no motion of the laboratory (experimental setup) relative to the aether. They detected no variation in velocity of light in moving and rest frames of reference. This was the first evidence of non-existence of ether. So the idea of the aether was rejected.

They performed an experiment by sending light signals simultaneously in two directions perpendicular to each other. Perpendicular directions are chosen to measure the variation in velocity of light due to relative motion of aether wind. It was supposed that there exist a gas aether in space which blow from one point to other point and transfer light just like wind blow due to gas in atmosphere.

A light ray coming from the source S is made incident on half-silvered mirror M . Half of the light is transmitted to mirror M_1 and half of the light is reflected toward mirror M_2. Light reflect back from M_1 and M_2 and recombines to form an interference pattern that the observer observes at point O (Fig.1.4). The two time intervals for light travel towards mirror M_1 and M_2 are (t_1, t_2) which differ slightly due to motion of Earth with velocity v. The two light rays will be slightly out of phase and interference will occur.

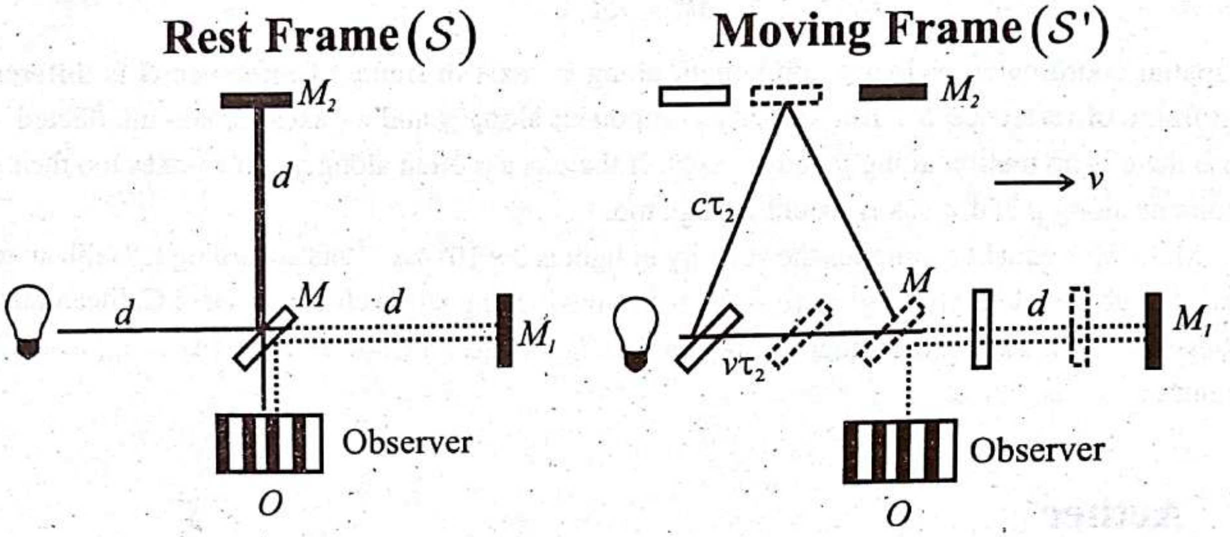

Figure 1.4: *Michelson-Morley Experiment at rest frame and moving frame*

Let c is the velocity of light through aether and d is the length of the perpendicular paths adopted by light then time taken by light from mirror M to M_1 will be

$$\begin{aligned} t_1 &= \frac{d}{c-v} + \frac{d}{c+v} \\ &= d\left[\frac{c+v+c-v}{(c-v)(c+v)}\right] \\ t_1 &= \frac{2cd}{c^2-v^2} \qquad (1.1) \end{aligned}$$

$(c-v)$ is the velocity when light moves towards mirror M_1 while $(c+v)$ is the velocity when light come back from mirror M_1

When light goes from mirror M to M_2 meanwhile mirror M_2 moves to right through a distance $v\tau_2$ and light travels a distance $c\tau_2$ relative to the aether. Then the time taken for this trip using right angle triangle (Fig.1.5) will be

$$\begin{aligned} (c\tau_2)^2 &= (v\tau_2)^2 + d^2 \\ \tau_2 &= \frac{d}{\sqrt{c^2-v^2}} \end{aligned}$$

Light reflects back from mirror M_2 to M (new position) and takes again the same time τ_2 to reach mirror

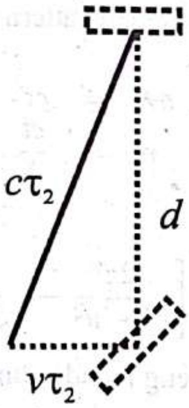

Figure 1.5: *Right angle triangle for light trip in moving frame*

M. The total time (t_2) taken by the light for Mirror M_2 will be

$$t_2 = 2\tau_2 = \frac{2d}{\sqrt{c^2 - v^2}} \tag{1.2}$$

To find time difference between path 1 (from mirror M_1) and 2 (from Mirror M_2), subtracting Eq.(1.1) from Eq.(1.2)

$$\begin{aligned} t_2 - t_1 &= \frac{2d}{\sqrt{c^2 - v^2}} - \frac{2cd}{c^2 - v^2} \\ \Delta t &= 2d\left[\frac{1}{\sqrt{c^2 - v^2}} - \frac{c}{c^2 - v^2}\right] \end{aligned} \tag{1.3}$$

whole arrangement of the experiment is rotated through 90° because velocity of aether (if exist) would change along path 1 and path 2. Then new time intervals due to rotation will be

$$\bar{t}_1 = \frac{2d}{\sqrt{c^2 - v^2}}, \qquad \bar{t}_2 = \frac{2cd}{c^2 - v^2}$$

So that

$$\begin{aligned} \Delta\bar{t} &= \bar{t}_2 - \bar{t}_1 = \frac{2cd}{c^2 - v^2} - \frac{2d}{\sqrt{c^2 - v^2}} \\ \Delta\bar{t} &= \bar{t}_2 - \bar{t}_1 = 2d\left[\frac{c}{c^2 - v^2} - \frac{1}{\sqrt{c^2 - v^2}}\right] \end{aligned} \tag{1.4}$$

Now using Eq.(1.3) and (1.4), the time difference between parallel and perpendicular directions will be

$$\begin{aligned} t &= \Delta\bar{t} - \Delta t \\ &= 2d\left[\frac{c}{c^2 - v^2} - \frac{1}{\sqrt{c^2 - v^2}}\right] - 2d\left[\frac{1}{\sqrt{c^2 - v^2}} - \frac{c}{c^2 - v^2}\right] \\ t &= 2d\left[\frac{2c}{c^2 - v^2} - \frac{2}{\sqrt{c^2 - v^2}}\right] \end{aligned} \tag{1.5}$$

The optical path difference produces an interference pattern with a shift of n fringes. i.e.,

$$\begin{aligned} n\lambda &= ct \\ n &= \frac{ct}{\lambda} \end{aligned}$$

Using Eq.(1.5)

$$n = \frac{2cd}{\lambda}\left[\frac{2c}{c^2 - v^2} - \frac{2}{\sqrt{c^2 - v^2}}\right]$$

In this experiment the effective length, wavelength and orbital velocity of Earth are

$$d = 11m, \quad \lambda = 5.9 \times 10^{-7}m \quad \text{and} \quad v = 3 \times 10^4 ms^{-1}$$

respectively. So that

$$n = 0.37$$

The Michelson-Morley interferometer can detect a minimum optical shift of 0.005 fringes. A fringe pattern shift was expected in the interference pattern but no fringe shift in the interference pattern was observed. So there is no effect on the velocity of light due to Earth's motion through the aether was found.

Results of Michelson-Morley experiment

Now, we summarize the results drawn from Michelson-Morley experiment.

- Galilean transformations are not valid at high velocity (valid only for mechanics at $v \ll c$ while invalid for electromagnetism at $v = c$).
- Velocity of light is constant in all inertial frames of reference.
- There is no absolute frame of reference such as aether.
- Law of physics (Newton's laws of motion) are same in all inertial frames of reference. This is called **principle of covariance of physical laws.**
- No experiment can detect the motion of inertial frame of reference in which it is performed called negative result of Michelson-Morley experiment.

1.5 Postulates of Special Theory of Relativity

Q. State and explain postulates of special theory of relativity.

Einstein found that equations of Newtonian mechanics were invariant under Galilean transformations whereas Maxwell's equations were invariant under Lorentz transformations. The invariance of velocity of light[7] required that either Maxwell's equations or Newtonian mechanics equations were to be modified. Einstein chosen Newtonian mechanics. He considered relative motion in inertial frames of reference where no forces act on the object and observers move relative to each other with uniform velocities. In this way he supported the idea of invariance of velocity of light.

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[7] Maxwell's equations predicted the velocity of electromagnetic radiation $c = 3 \times 10^8 ms^{-1}$. This is actually upper velocity limit of any object.

In 1905, Einstein formulated special theory of relativity in his remarkable paper "on the Electrodynamics of moving bodies" based on two postulates. The entire special theory of relativity is based on two assumptions but this theory is also consistent with experiments. This theory also redefines the terms space, time and simultaneity.

1.5.1 The Principle of Relativity or Equivalence

The laws of physics are invariant in all inertial frames.

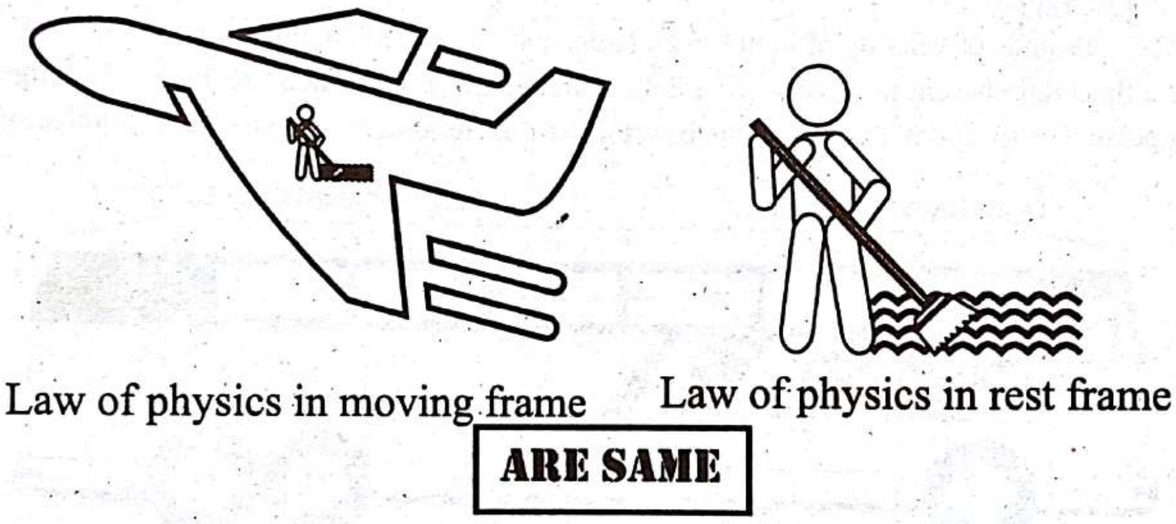
Law of physics in moving frame
Law of physics in rest frame
ARE SAME

Figure 1.6: *The two observers shown are in different inertial frames of reference. According to the first postulate of relativity, physical experiments will give identical laws of nature in the two frames.*

Galilean transformations implies that all inertial observers will agree about laws of Newtonian mechanics (Fig:1.6). This is the first theory of relativity called Galilean relativity. Einstein found a drawback in Galileo's theory of relativity. He believed that inertial observers would agree about the laws of physics generally, not just in mechanics.

Einstein presumed that velocity of light must be same in vacuum for all inertial observers. This idea led him to develop new theory of relativity based on Maxwell's equations of electromagnetism. This theory caused laws of mechanics to be rewritten.

This theory also has impact on thermodynamics, electromagnetism, optics, nuclear physics and high-energy physics. This theory is only applicable to inertial frames of reference, that's why it is called special theory of relativity.

1.5.2 The principle of constancy of the velocity of light

The velocity of light in free space has the same value in all inertial frames of reference.

According to Galilean velocity transformations, velocity of light must vary due to relative motion between different frames of reference. This postulate is about the invariance of velocity of light in all inertial frames of reference which contradicts Galilean transformations.

In our daily life experience velocity of two objects in relative motion is either $(v+u)$ or $(v-u)$ that depends on their direction of motion. In the same way for light, the velocity must be either $(c+u)$ or $(c-u)$ but it is not true.

This postulate also accounts for the results of Michelson-Morley experiment about the invariance of the velocity of light.

The constancy of velocity of light can be understood by following illustration:
Suppose a flash light is sent to a driver from light source in the bus. At the same time a flash light is also sent to a passenger sitting at the back of the bus (Fig:1.6). If the velocity of bus is u then velocity of flash light must be $(c-u)$ for driver and $(c+u)$ for the passenger at the back of the bus according to Galilean velocity transformations. But the driver and the passenger detect the velocity of light only c. This may not agree with common sense but it is true. The velocity of light was observed to be c which shows that velocity of light is invariant.

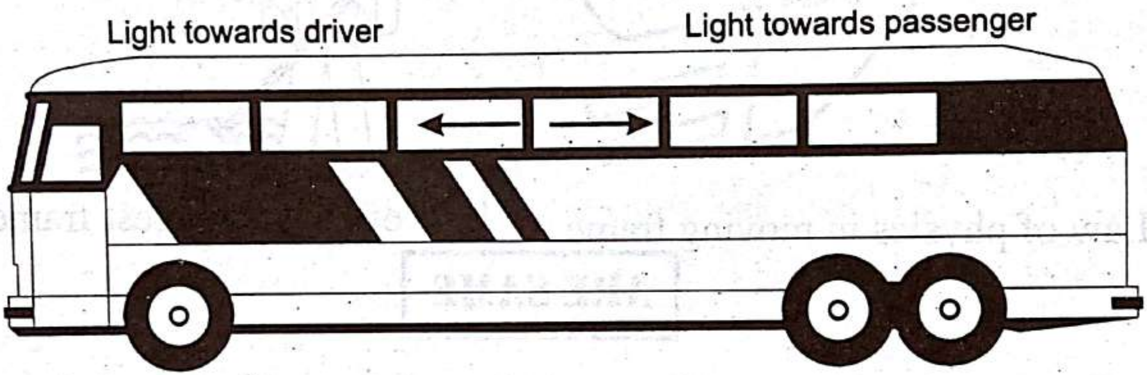

Figure 1.7: *A flash of light is sent out by a person in a moving bus towards driver and passenger at the back.*

There are two requirements of the special theory of relativity:

- The correct transformation between inertial frames which keeps the velocity of light constant.
- To check the consistency of physical laws concerning the transformation.

1.6 The Lorentz Transformations

Q. What are Lorentz transformations equations? Derive its equations for space and time coordinates. Also show that Lorentz transformations equations reduce to Galilean transformation equations for $v \ll c$

The transformations of spacetime coordinates between two frames of reference that move at constant velocity relative to each other are called Lorentz transformations.

Lorentz[8] derived the Lorentz transformations in order to explain the results of the Michelson-Morley experiment and replaced Galilean transformations equations. These transformation equations are not only applicable in mechanics but also in electromagnetism, optics, thermodynamics and other areas of physics.

Lorentz transformations are also consistent with the invariance of velocity of light but were derived before the development of special theory of relativity.

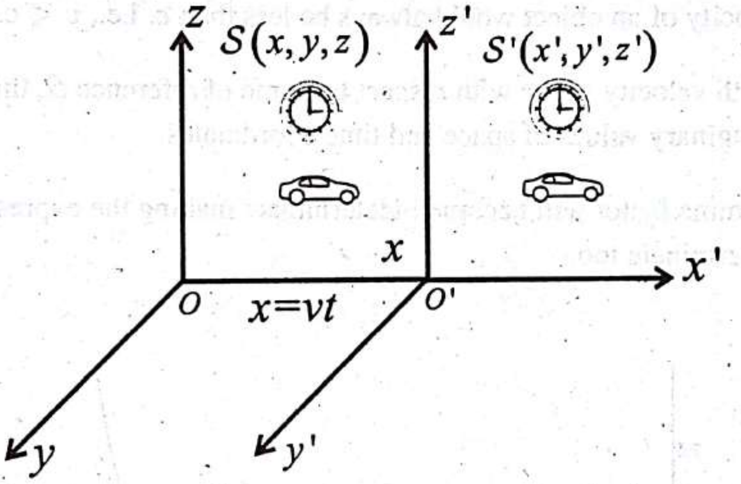

Figure 1.8: *An event in different Coordinate Systems in relative motion*

Let us consider in a frame of reference S an observer O is observing an event (Fig.1.8), then in another frame of reference S' moving with constant velocity v along positive x-axis O' will observe the same event with different space and time coordinates. The relative space and time coordinates will be related by Lorentz transformation equations given as

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right) \end{aligned}$$

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[8] Hendrick Lorentz (1853-1928) was a Dutch physicist who was awarded 1902 Nobel prize for the discovery of Zeeman effect. He also derived coordinate transformation equations in special theory of relativity.

where

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

called **Lorentz factor** or **gamma factor**. Its value is

$$1 \leq \gamma < \infty$$

and is a function of velocity (Fig.1.9). The value of velocity is $-c < v < c$. Negative velocity is along negative $x-$axis.

Lorentz transformations and velocity of objects

Lorentz transformations impose a condition on the velocity of objects. There are three possibilities of velocity of objects:

- The physical velocity of an object would always be less than c. i.e., $v < c$.
- If it is moving with velocity $v > c$ with respect to frame of reference S, the Lorentz transformation would lead to imaginary values of space and time coordinates.
- For $v = c$, the gamma factor will become indeterminate making the expressions for space and time coordinates indeterminate too.

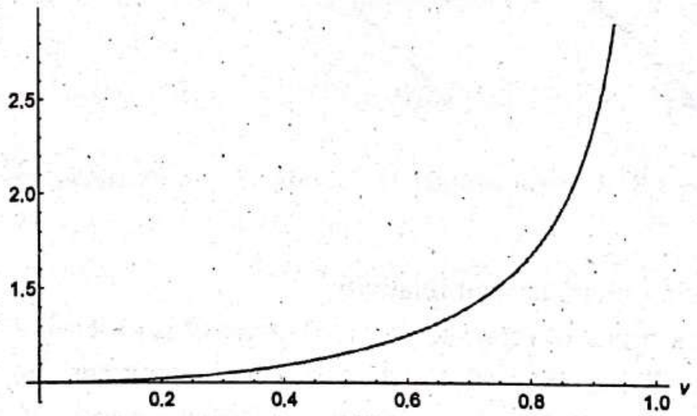


Figure 1.9: *Gamma factor as a function of velocity*

Hence in special theory of relativity, the velocity of light is not only a universal constant but is also the upper limit on all physical velocities (e.g., group velocity). However, special theory of relativity does not prevent us from considering geometrical velocities (e.g., phase velocity) exceeding the velocity of light. It only asserts that data or information (e.g., group velocity) cannot be transmitted with a velocity greater than velocity of light.

A rotation free Lorentz transformation is called a **Lorentz boost**. Boost is something different from simple displacement. There is no relative motion in case of displacement but boost involves relative motion.

Another form of Lorentz transformations can be written using gamma factor γ in terms of velocity parameter $\beta = \frac{v}{c}$ as

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}}$$

so that

$$\begin{aligned} x' &= \frac{x - vt}{\sqrt{1-\beta^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - \frac{\beta x}{c}}{\sqrt{1-\beta^2}} \end{aligned}$$

Lorentz transformations replaced the Galilean transformations. These equations were modified by Poincaré to include translations.

1.6.1 Derivation of the Lorentz Transformations

Let us consider an event taking place (Fig.1.8) is being observed in a frame of reference S at coordinates (x, y, z, t). In another frame of reference S' moving with constant relative velocity v along positive x-axis the same event will be observed at coordinates (x', y', z', t'). Here, we find a functional relationships in isotropic and homogeneous space

$$\begin{aligned} x' &= x'(x, y, z, t) \\ y' &= y'(x, y, z, t) \\ z' &= z'(x, y, z, t) \\ t' &= t'(x, y, z, t). \end{aligned}$$

So there is a (1-1) correspondence between the points of S and S'. i.e., the space and time coordinates (x', y', z', t') and (x, y, z, t) will be related by linear transformation equations.

At $t = t' = 0$ origins $\mathcal{O}$ and $\mathcal{O}'$ coincide. i.e., coordinate systems are synchronized. The transformation equations for observing the event in S' frame will be of the form

$$\begin{aligned} x' &= a_1 x + a_2 y + a_3 z + a_4 t \\ y' &= b_1 x + b_2 y + b_3 z + b_4 t \\ z' &= c_1 x + c_2 y + c_3 z + c_4 t \\ t' &= d_1 x + d_2 y + d_3 z + d_4 t \end{aligned}$$

Where a, b, c and d are coefficients which are to be determined from exact transformation equations. Their values depend upon the relative velocity v of the two inertial frames. We intend to calculate the values of coefficients valid for any value of v $(v < c)$. As there is no relative motion along y and z-axes so

$b_2 = c_3 = 1$ and $b_1 = b_3 = b_4 = c_1 = c_2 = c_4 = 0$. Therefore above set of equations become

$$\begin{aligned} x' &= a_1x + a_2y + a_3z + a_4t && (1.6)\\ y' &= y && (1.7)\\ z' &= z && (1.8)\\ t' &= d_1x + d_2y + d_3z + d_4t && (1.9) \end{aligned}$$

Due to symmetric $yz-$plane assume that t' does not depend on y and z coordinates otherwise clocks symmetric in yz-plane must disagree as observed from frame of reference S' and it contradicts the isotropy of space. Therefore, putting $d_2 = d_3 = 0$ in Eq.(1.9)

$$t' = d_1x + d_4t \tag{1.10}$$

Using the fact that the origin of frame of reference S' frame is approachable from frame of reference S frame by covering some distance x in time t with velocity v. i.e., at $x = vt, x' = 0$. So Eq.(1.6) becomes

$$0 = a_1(vt) + a_2y + a_3z + a_4t$$

or

$$(0)t + (0)x + (0)y + (0)z = (a_1v + a_4)\,t + a_2y + a_3z$$

Comparing coefficients of t, y and z on both sides

$$\begin{aligned} 0 &= a_1v + a_4 \qquad \Rightarrow a_1 = -a_4v\\ a_2 &= 0\\ a_3 &= 0 \end{aligned}$$

Putting above values in Eq.(1.6)

$$x' = a_1x - a_1vt = a_1(x - vt) \tag{1.11}$$

Using the principle of the constancy of velocity of light with spherical wavefront at O (Fig.1.10),

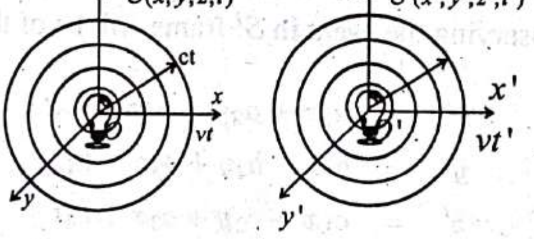


Figure 1.10: *Inertial and non-inertial frame of reference*

$$x^2 + y^2 + z^2 = c^2t^2 \tag{1.12}$$

$$x'^2 + y'^2 + z'^2 = c^2 t'^2$$

Using values from Eqns.(1.7), (1.8), (1.10) and (1.11)

$$\begin{aligned} \left[a_1 (x - vt)\right]^2 + y^2 + z^2 &= c^2 (d_1 x + d_4 t)^2 \\ a_1^2 \left(x^2 + v^2 t^2 - 2xvt\right) + y^2 + z^2 &= c^2 \left(d_4^2 x^2 + d_4^2 t^2 + 2 d_1 d_4 x t\right) \end{aligned}$$

Rearranging,

$$\left(a_1^2 - c^2 d_1^2\right) x^2 + y^2 + z^2 - 2\left(a_1^2 v + d_1 d_4 c^2\right) xt = \left(c^2 d_4^2 - a_1^2 v^2\right) t^2$$

Comparing coefficients of x^2, xt and t^2 with that of Eq.(1.12)

$$a_1^2 - c^2 d_1^2 = 1 \qquad a_1^2 v + d_1 d_4 c^2 = 0 \qquad c^2 d_4^2 - a_1^2 v^2 = c^2$$

Solving these equations simultaneously for coefficients, we have

$$a_1 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma, \qquad d_1 = -\frac{v/c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = -\frac{\gamma v}{c^2}, \qquad d_4 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma$$

Putting these values in Eq.(1.7), (1.8), (1.10) and (1.11), we have

$$\begin{aligned} x' &= \gamma (x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma \left(t - \frac{v}{c^2} x\right) \end{aligned}$$

These are called **Lorentz transformation equations.**

1.6.2 Inverse Lorentz Transformations

The transformations equations used to transform coordinates in frame of reference S' back into frame of reference S are called inverse Lorentz transformations.

Lorentz transformations can also be written for an event relating frame of reference S with frame of reference S'. The Lorentz transformation equations can be solved to give inverse Lorentz transformations

$$\begin{aligned} x &= \gamma (x' + vt) \\ y &= y' \\ z &= z' \\ t &= \gamma \left(t' + \frac{v}{c^2} x'\right) \end{aligned}$$

These transformations are identical in form with Lorentz transformations if v is replaced with $-v$ and primed quantities are interchanged with unprimed quantities.

1.6.3 Interval Coordinate Transformations

> The difference in the corresponding space and time coordinates of the two events is called interval.

An interval between two events measured along a specific axis in a given frame of reference can be determined using Lorentz transformation (Table: 1.1). Interval coordinate transformations can be written for two events in terms of time and space coordinates as (Table: 1.2) To get coordinate intervals, subtract equations for event 1 from that of event 2 along $x-$axis

Event 1	Event 2
$x_1' = \gamma\left(x_1 - vt_1\right)$	$x_2' = \gamma\left(x_2 - vt_2\right)$
$y_1' = y_1$	$y_2' = y_2$
$z_1' = z_1$	$z_2' = z_2$
$t_1' = \gamma\left(t_1 - \frac{v}{c^2}x_1\right)$	$t_2' = \gamma\left(t_2 - \frac{v}{c^2}x_2\right)$

Table 1.1: *Lorentz transformations for two events*

$$\begin{aligned}
\Delta x' &= x_2' - x_1' \\
&= \gamma(x_2 - vt_2) - \gamma(x_1 - vt_1) \\
&= \gamma\left[(x_2 - x_1) - v(t_2 - t_1)\right] \\
\Delta x' &= \gamma(\Delta x - v\Delta t)
\end{aligned}$$

Similarly along $y-$axis

$$\begin{aligned}
\Delta y' &= y_2' - y_1' \\
&= y_2 - y_1 \\
\Delta y' &= \Delta y
\end{aligned}$$

and along $z-$axis

$$\begin{aligned}
\Delta z' &= z_2' - z_1' \\
&= z_2 - z_1 \\
\Delta z' &= \Delta z
\end{aligned}$$

and time interval

$$\begin{aligned}
\Delta t' &= t_2' - t_1' \\
&= \gamma\left(t_2 - \frac{v}{c^2}x_2\right) - \gamma\left(t_1 - \frac{v}{c^2}x_1\right)
\end{aligned}$$

$$\begin{aligned} &= \gamma\left[t_2 - t_1 - \frac{v}{c^2}(x_2 - x_1)\right] \\ \Delta t' &= \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) \end{aligned}$$

The inverse Lorentz interval transformation can also be obtained by replacing v with $-v$ and interchanging primed and unprimed quantities in above equations

$$\begin{aligned} \Delta x &= \gamma(\Delta x' + v\Delta t) \\ \Delta y &= \Delta y' \\ \Delta z &= \Delta z' \\ \Delta t &= \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right) \end{aligned}$$

These interval transformations are useful because in calculations or experiments length or time between two points is the main concern. Lorentz interval transformations and inverse interval transformations are summarized in the table (1.2)

Interval transformations	**Inverse interval transformations**
$\Delta x' = \gamma(\Delta x - v\Delta t)$	$\Delta x = \gamma(\Delta x' + v\Delta t')$
$\Delta y' = \Delta y$	$\Delta y = \Delta y'$
$\Delta z' = \Delta z$	$\Delta z = \Delta z'$
$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)$	$\Delta t = \gamma\left(\Delta t + \frac{v}{c^2}\Delta x\right)$

Table 1.2: *Lorentz interval and inverse interval transformations*

1.7 Consequences of Lorentz Transformations

Write a note on consequences of Lorentz transformations. Lorentz transformations not only replaced Galilean transformations but also has startling predictions. Some of the consequences of Lorentz transformations are discussed below.

1.7.1 Time Dilation

The phenomenon that there is a difference in the elapsed time measured by two observers in relative motion is called time dilation.

Let us consider an event of flash light is being observed in two frames of reference $S(x, y, z, t)$ and $S'(x', y', z', t')$ in relative motion with each other. A flash light is sent from light source towards a receiver. The separation between the light source and receiver is ℓ as seen from the rest frame of reference S. Flash of light covers a total distance of 2ℓ as shown in Fig(1.11). The time taken by flash light for the journey as measured by the clock in rest frame of reference will be

$$t = \frac{S}{v} = \frac{2\ell}{c} \tag{1.13}$$

When same event is observed in frame of reference S' moving with velocity v, it is seen that flash light makes a longer path covering distance $2\ell'$. Then from the geometry of Fig (1.11)

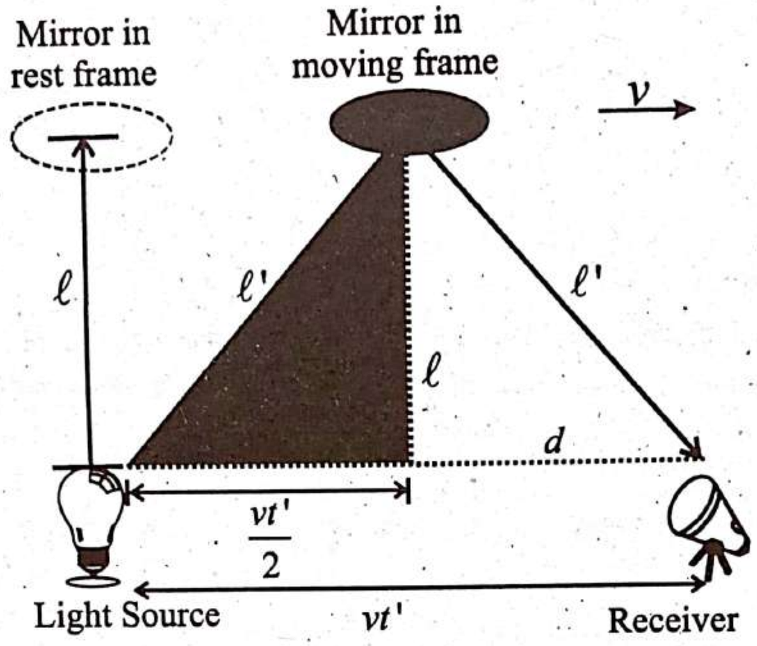

Mirror in rest frame
Mirror in moving frame
ν
ℓ
ℓ'
ℓ'
ℓ
d
vt'/2
Light Source
vt'
Receiver

Figure 1.11: *Light ray reflecting from a mirror in rest and moving frame*

$$\begin{aligned}
\ell'^2 &= d^2 + \ell^2 \\
\left(\frac{ct'}{2}\right)^2 &= \left(\frac{vt'}{2}\right)^2 + \ell^2 \\
(c^2 - v^2)\frac{t'^2}{4} &= \ell^2 \\
t' &= \frac{2\ell}{c\sqrt{1 - \frac{v^2}{c^2}}}
\end{aligned}$$

Using Eq.(1.13)

$$\begin{aligned}
t' &= \frac{t}{\sqrt{1 - \frac{v^2}{c^2}}} \\
t' &= \gamma t
\end{aligned}$$

For $v \ll c, \gamma \to 1$ so that

$$t' \approx t$$

Thus time dilation phenomenon is not observable for velocity very less than velocity of light (Fig.1.12).

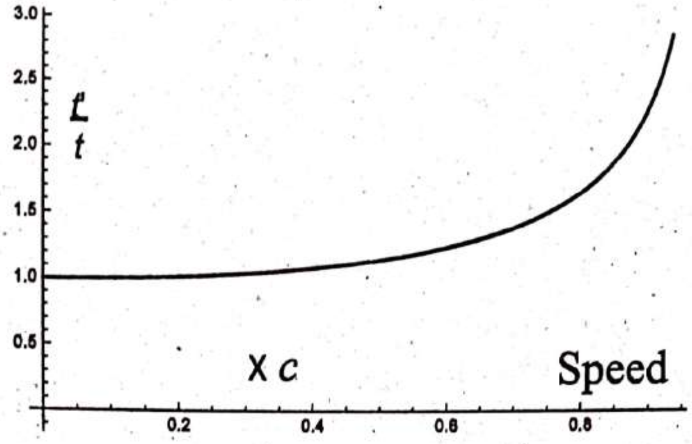


Figure 1.12: *Time Dilation as a function of velocity*

The time interval between two events in rest frame is called **proper time.** Thus above equation says that time in moving frame passes slow than in rest frame. It also means that time intervals between two events is longer in moving frame than in rest frame. This effect is called **time dilation.**

It is not an illusion but it is a real phenomenon widely observed in high-energy physics. For example, lifetime of muon in laboratory frame is only $2.2\mu s$ whereas the lifetime of muon produced by cosmic ray (velocity equal to 98% of velocity of light) is $16\mu s$. This is about five times longer the time at rest frame.

The Twin Paradox

A thought experiment involving twin persons in relative motion in space time is called twin paradox.

Suppose twins A and B are on the earth and A takes off in a spaceship moving at the constant velocity of $0.9c$ for a space voyage at 21st birthday.

After 20 years A comes back and rejoin twin B who stayed at home on the Earth. He was shocked to see twin B because twin B was aged more than his age.

In principle both A and B were in relative motion and going away from each other with same velocity. So each should feel time dilation paradoxically. But the situation was quite different. This is a logical contradiction in their relative motions. Einstein considered this difference as natural consequence of special relativity. Some scientists argued that the age difference is due to gravitational time dilation.

As viewed from earth the moving clock (A's clock) has been running slow according to time dilation by a factor γ given by

$$t' = \frac{t}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t' = \frac{10}{\sqrt{1-\frac{(0.9c)^2}{c^2}}} = 23years$$

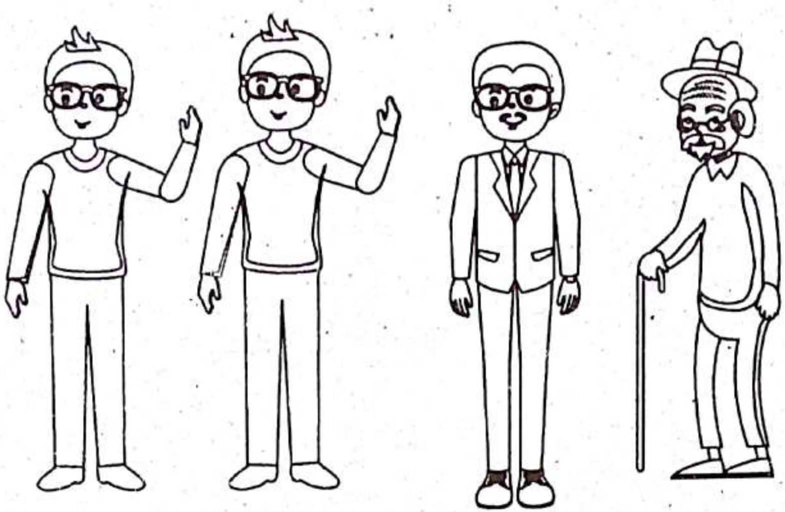


Figure 1.13: *Twins A and B before and after space voyage*

Similarly on return time elapsed is same. So total time elapsed as of Earth time is

$$t'_{\text{total}} = 2t' = 2(23) = 46years$$

Thus the trip will require 20 years according to A's clock but as of B's clock the time required for the trip is 46years. Therefore B will be celebrating his 67th (21+46) birthday. This is called **twin paradox**[9].

The twin paradox is also experimentally verified using atomic clocks by measuring time dilation between flights of aircraft and satellites. Another example of experimental verification is particles accelerator used to measure time dilation of circulating particle beams.

1.7.2 Length Contraction

A phenomenon that a moving object's length is measured to be shorter than its proper length is called length contraction.

The relativistic effect that relates the length of an object observed from different frames of reference was suggested by Lorentz and FitzGerald and also known as Lorentz-Fitzgerald contraction.

To develop a relationship between lengths that are measured parallel to the direction of motion in various coordinate systems, we consider another thought experiment. We attach a light source to one end of a ruler and a mirror to the other end. The ruler is at rest in reference frame S, and its length in this frame is ℓ as shwon in Fig(1.11). Then the time t required for a light pulse to make the round trip from source to mirror and back is.

$$t = \frac{S}{v} = \frac{2\ell}{c}$$

This is a proper time interval because departure and return occur at the same point in S'.

In reference frame S the ruler is moving to the right with speed v during this travel of the light pulse. The length of the ruler in S is ℓ', and the time of travel from source to mirror, as measured in S, is

---

[9] Paradox is a situation that seems to be impossible but is actually possible.

t_1 . During this interval the ruler, with source and mirror attached, moves a distance vt_1. The total length of path d from source to mirror is not ℓ', but rather

$$d = \ell' + vt_1 \tag{1.14}$$

The light pulse travels with speed c, so it is also true that

$$d = ct_1 \tag{1.15}$$

Putting Eq.(1.15) in Eq.(1.14)

$$ct_1 = \ell' + vt_1$$

or

$$t_1 = \frac{\ell'}{c - v}$$

In the same way we can show that the time t_2 for the return trip from mirror to source is

$$t_2 = \frac{\ell'}{c + v}$$

To find total time $t' = t_1 + t_2$ adding above two equations

$$t' = \frac{\ell'}{c - v} + \frac{\ell'}{c + v} = \frac{2\ell'}{c\left(1 - \frac{v^2}{c^2}\right)} \tag{1.16}$$

Time dilation is given by

$$t' = \frac{t}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{2\ell}{c\sqrt{1 - \frac{v^2}{c^2}}} \tag{1.17}$$

Comparing Eq.(1.16) and Eq.(1.17)

$$\frac{2\ell'}{c\left(1 - \frac{v^2}{c^2}\right)} = \frac{2\ell}{c\sqrt{1 - \frac{v^2}{c^2}}}$$

or

$$\ell' = \ell\sqrt{1 - \frac{v^2}{c^2}}$$

For $v \ll c, \gamma \to 1$ so that

$$\ell' \approx \ell$$

Thus length contraction phenomenon is not observable for velocity very less than velocity of light (Fig:1.14).

Above equation says that length in moving frame of reference becomes shorter. It also means that distance between initial and final positions become shorter as seen from moving frame of reference. This effect is called **length contraction.**

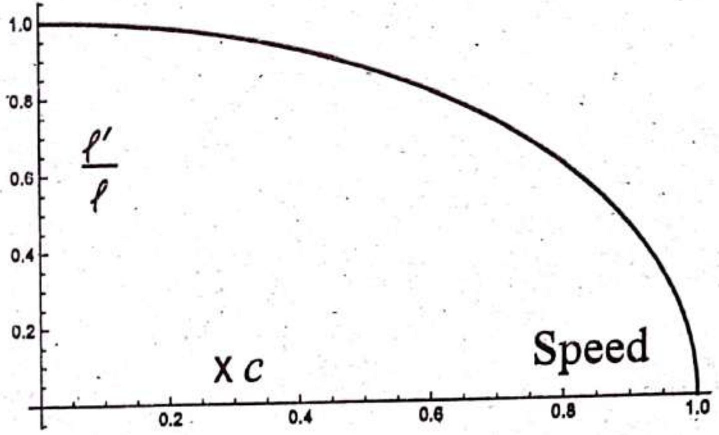


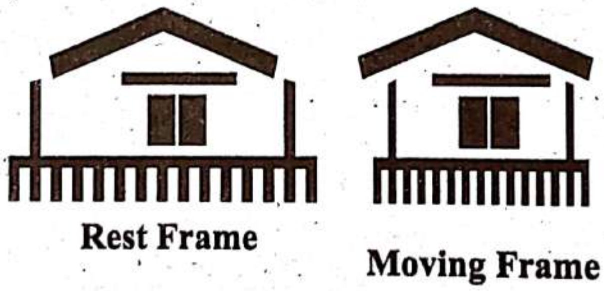
Figure 1.14: *Length contraction as a function of velocity*

1.7.3 Ladder Paradox

A thought experiment involving a ladder in horizontal relative motion in space and time is called ladder paradox.

Suppose a ladder is to be fitted in a small room of length ℓ. The thought experiment is done in two different frames of reference which are in relative motion. When the ladder is placed in room in rest frame of reference S, the room space was too small for the ladder. On the other hand, when the ladder is placed in moving room in frame of reference S' the room was enough to fit the ladder(Fig.1.15).

As the objects move they face length contraction. It is also called Pole-Barn paradox.



Rest Frame

Moving Frame

Figure 1.15: *Ladder in a room in rest and moving frame*

1.7.4 Relativity of Simultaneity

Events that occur at the same time in one frame of reference do not necessarily occur at the same time in another frame of reference. This is called relativity of simultaneity.

Suppose two events A and B happen simultaneously in frame of reference S at $t_1 = t_2$ then they will happen in moving frame S' at t'_1 and t'_2 given by

$$t'_1 = \gamma\left(t_1 - \frac{v}{c^2}x_1\right) \qquad t'_2 = \gamma\left(t_2 - \frac{v}{c^2}x_2\right)$$

The time interval between these events will be

$$t_2' - t_1' = \gamma\left(t_2 - \frac{v}{c^2}x_2\right) - \gamma\left(t_1 - \frac{v}{c^2}x_1\right)$$

As events are simultaneous[10] in rest frame i.e., $t_1 = t_2$ then

$$\begin{aligned} t_2' - t_1' &= \gamma\left(t_1 - \frac{v}{c^2}x_2\right) - \gamma\left(t_1 - \frac{v}{c^2}x_1\right) \\ t_2' - t_1' &= -\gamma\frac{v}{c^2}\left(x_2 - x_1\right) \neq 0 \end{aligned}$$

So events are not simultaneous in moving frame. There maybe a particular case in which two events simultaneous in rest frame are also simultaneous in moving frame. For example, two events happening in different frames of reference having zero relative velocity.

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[10] Event simultaneous in rest frame are not necessarily to be simultaneous in moving frame.

KEY IDEAS

- Relativity means that it is physically impossible to detect absolute motion. This can be stated as a symmetry in physics: physics equations are unchanged under transformations among reference frames in relative motion.
- Special relativity (SR) is the symmetry with respect to coordinate transformations among inertial frames, general relativity (GR) the symmetry among more general frames, including accelerating coordinate systems or systems in which gravity is present.
- Tensor equations are covariant under coordinate transformation. The familiar case of rotation symmetry illustrates how coordinate symmetry follows automatically when the equations of physics are written in tensor form.
- Newtonian mechanics is covariant under Galilean transformations among inertial frames of reference. Maxwell's equations lack this Galilean symmetry.
- The apparent contradiction between constancy of light velocity and classical (i.e., Galilean and Newtonian) relativity led most physicists to assume the presence of a unique reference frame, an electromagnetic ether. Einstein took a different approach by developing a new conception of time, leading to a completely new kinematics. Although both viewpoints would lead independently to the Lorentz transformation, their physics contents are fundamentally different.
- The equivalence principle between acceleration and gravity played an important role in Einstein's progress from SR to GR, which is a geometric theory of gravitation.

KEY TERMS

- **Frame of Reference:** A laboratory or place where some phenomena are observed and measurements are taken is called frame of reference.
- **Inertial Frame of Reference:** The coordinate system which is moving with uniform velocity or at rest is called inertial frame of reference.
- **Non-Inertial Frame of Reference:** The coordinate system which is moving with variable velocity is called non-inertial frame of reference
- **Observer:** The person who examines an event in a particular frame of reference is called observer.
- **Event:** An instantaneous occurrence taking place in space at a specific time is called an event.
- **Coordinate Transformations:** The set of equations which relate space and time coordinates in different frames of reference is called coordinate transformations.

- **Galilean Transformation:** Galilean transformations are coordinate transformations, in classical physics, which connects one inertial frame of reference with another inertial frame of reference.
- **Aether:** The hypothetical substance through which electromagnetic waves travel is called aether.
- **Gamma Factor:** The factor by which length, time or mass of an object, in relative motion, changes is called gamma factor.
- **Lorentz Boost:** A rotation free Lorentz transformation is called a Lorentz boost.
- **Time Dilation:** The phenomenon that there is a difference in the elapsed time measured by two observers in relative motion is called time dilation.
- **Proper Time:** The time interval between two events in rest frame is called proper time.
- **Twin Paradox:** A thought experiment involving twin persons in relative motion in space time is called twin paradox.
- **Length Contraction:** A phenomenon that a moving object's length is measured to be shorter than its proper length is called length contraction.
- **Proper Length:** The length of an object in rest frame is called proper length.
- **Ladder Paradox:** A thought experiment involving a ladder in horizontal relative motion in space time is called ladder paradox.

KEY FORMULAE

Galilean transformation

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}$$

Relative velocity

$$\begin{aligned} u'_x &= u_x \\ u'_y &= u_y \\ u'_z &= u_z \end{aligned}$$

Lorentz transformation

$$\begin{aligned} x' &= \gamma(x - vt) = \frac{x - vt}{\sqrt{1-\beta^2}} \\ y' &= y \end{aligned}$$

$$
\begin{aligned}
z' &= z \\
t' &= \gamma\left(t - \frac{v}{c^2}x\right) = \frac{t - \dfrac{\beta x}{c}}{\sqrt{1-\beta^2}}
\end{aligned}
$$

Gamma factor

$$\gamma = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

Inverse Lorentz transformation

$$
\begin{aligned}
x &= \gamma\,(x' + vt) \\
y &= y' \\
z &= z' \\
t &= \gamma\left(t' + \frac{v}{c^2}x'\right)
\end{aligned}
$$

Interval transformation

$$
\begin{aligned}
\Delta x' &= \gamma\,(\Delta x - v\Delta t) \\
\Delta y' &= \Delta y \\
\Delta z' &= \Delta z \\
\Delta t' &= \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)
\end{aligned}
$$

Inverse interval transformation

$$
\begin{aligned}
\Delta xe &= \gamma\,(\Delta x' + v\Delta t') \\
\Delta y &= \Delta y' \\
\Delta z &= \Delta z' \\
\Delta t &= \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right)
\end{aligned}
$$

Time dilation

$$t' = t$$

Length contraction

$$\ell' = \gamma\ell$$

EXERCISE

Problems

1. Prove that for $v \ll c$, Lorentz transformation equations reduce to classical Galilean transformation equations.

2. In muon decay, the lifetime in rest frame of reference is $2.2\mu s$ and the distance travelled by it with velocity $0.98c$ relative to rest frame of reference is $646.8m$. Determine the following in moving frame of reference
   (a) length contraction
   (b) time dilation

3. If somehow the relative velocity of frames of reference is $v > c$, how
   (a) Galilean transformations are modified?
   (b) Lorentz transformations are modified?

4. Write the Lorentz transformation equations in matrix form in terms of
   (a) Lorentz factor
   (b) velocity parameter

   Also write the inverse Lorentz transformation equations by determining the inverse Lorentz transformation matrix.

5. A train $100m$ long is stationery at a platform. When it is travelling at the velocity of $0.8c$ and passes through a tunnel $80m$ long what will be the length of the train.

6. Find the Lorentz factor for
   (a) $v = 0.1c$
   (b) $v = 0.8c$

7. Two particles are moving with velocities u and v in a frame of reference. If their relative velocity is w then prove that the Lorentz factor $\gamma(w)$ is

   $$\gamma(w) = \gamma(u)\gamma(v)\left(1 - \frac{u \cdot v}{c^2}\right).$$

   Also find the relation for relative velocity w.

8. The matrix equation

   $$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma/c \\ -\gamma/c & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

   can be inverted to determine the coordinates (ct, x) in terms of (ct', x'). Show that inverting the 2×2 matrix leads to the inverse Lorentz transformations.

9. Suppose a clock moving with uniform velocity v relative to a clock at rest measure the time interval between two successive events in its own frame of reference to be . How would the time interval measured by the rest clock in its own frame of reference be related to . If the moving clock gets accelerated then what would be the form of the relation between two times if we use clock hypothesis?

Chapter 2

Relativistic Mechanics

Learning Objectives

In this chapter you will learn

- To derive velocity addition and Lorentz velocity transformations.
- To derive acceleration Lorentz transformations
- To derive momentum Lorentz transformations
- To derive force Lorentz transformations
- Relativistic Doppler effect and its cases.
- Abberation of light

KEY TERMS

velocity addition principle, relativistic Doppler effect, abberation of light

2.1 Velocity Addition and Transformation

Q. Derive relation for velocity addition and velocity Lorentz transformations.

Consider an object is moving with velocity $u(u_x, u_y, u_z)$ in an inertial frame S then its velocity in frame S' moving uniformly with velocity v will be $u'(u'_x, u'_y, u'_z)$(Fig:2.1). Then space and time coordinates in both frames are related by Galilean transformation as

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}$$

and corresponding velocity components are

$$\begin{aligned} \frac{dx'}{dt'} &= \frac{dx'}{dt} = u'_x = u_x - v \\ \frac{dy'}{dt'} &= \frac{dy'}{dt} = u'_y = u_y \\ \frac{dz'}{dt'} &= \frac{dz'}{dt} = u'_z = u_z \end{aligned}$$

These are called **Galilean velocity addition rules.** According to this rule speed of light must be either $(c+u)$ or $(c-u)$. But Michelson-Morley in their experiment predicted that speed of light is invariant. Here Galilean velocity addition formula was failed. So a new velocity addition formula was required to compensate the drawback of Galilean transformations which was settled by Lorentz transformation in special relativity. The necessary condition of the new formula was that speed of light should be same in all inertial frames.

According to Lorentz transformations

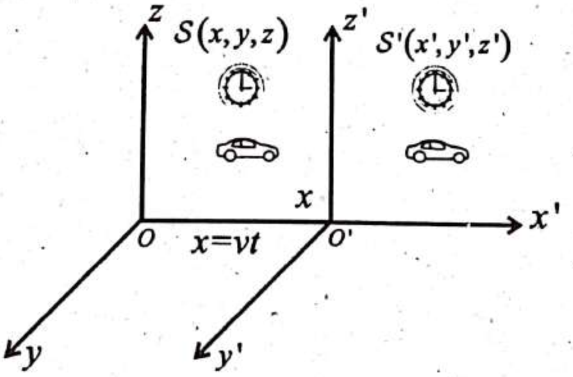


Figure 2.1: *Object moving in inertial frames of reference in the relative motion.*

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

Taking differentials of above equations

$$\begin{aligned} dx' &= \gamma(dx - vdt) \\ dy' &= dy \\ dz' &= dz \\ dt' &= \gamma\left(dt - \frac{v}{c^2}dx\right) \end{aligned}$$

Dividing the first three equations by the fourth equation

$$u'_x = \frac{dx'}{dt'} = \frac{\gamma(dx - vdt)}{\gamma\left(dt - \frac{v}{c^2}dx\right)} = \frac{dt\left(\frac{dx}{dt} - v\right)}{dt\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)} = \frac{u_x - v}{1 - \frac{v}{c^2}u_x} \tag{2.1}$$

$$u'_y = \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - \frac{v}{c^2}dx\right)} = \frac{dy}{\gamma dt\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)} = \frac{u_y}{\gamma\left(1 - \frac{v}{c^2}u_x\right)} \tag{2.2}$$

$$u'_z = \frac{dz'}{dt'} = \frac{dz}{\gamma\left(dt - \frac{v}{c^2}dx\right)} = \frac{dz}{\gamma dt\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)} = \frac{u_z}{\gamma\left(1 - \frac{v}{c^2}u_x\right)} \tag{2.3}$$

These expressions give Lorentz transformation for velocity components in frame S to those in frame S' in the terms of Cartesian coordinate system. These formulae are also known as velocity addition rule in special relativity.

To check the consistency of above equations with speed of light put $u_x = u = c$ and $u'_x = u'$ in Eq.(2.1) for a particle moving parallel to the x-axis. So that

$$\begin{aligned} u' &= \frac{c - v}{1 - \frac{cv}{c^2}} \\ &= \frac{c - v}{1 - \frac{v}{c}} \\ &= \frac{c(c - v)}{(c - v)} \\ u' &= c \end{aligned}$$

Thus a particle which is moving with velocity c observed by an observer in frame S and its velocity observed by S' is also c. The velocity of the frame S' has no effect on the velocity of light. This is a verification of the fact that Lorentz transformation is consistent with the postulate of the constancy of velocity of light and hence is in accordance with the results of Michelson-Morley experiment. It can be assumed that Galilean transformations are a special case of Lorentz transformations for $v \ll c$. In this case

Eqs.(2.1)-(2.3) becomes

$$\begin{aligned} u'_x &= u - v \\ u'_y &= u_y \\ u'_z &= u_z \end{aligned}$$

These are the classical **Galilean velocity addition rules**. Classical and relativistic velocity addition are compared in the fig(2.2)

The inverse Lorentz transformations for velocity can be obtained by replacing v with $-v$ and interchanging primed quantities with unprimed quantities (Eqns.2.1-2.3). So that

$$\begin{aligned} u_x &= \frac{u'_x + v}{1 + \frac{v}{c^2} u'_x} \\ u_y &= \frac{u'_y}{1 + \frac{v}{c^2} u'_x} \\ u_z &= \frac{u'_z}{1 + \frac{v}{c^2} u'_x} \end{aligned}$$

These are the inverse Lorentz transformations for velocity.

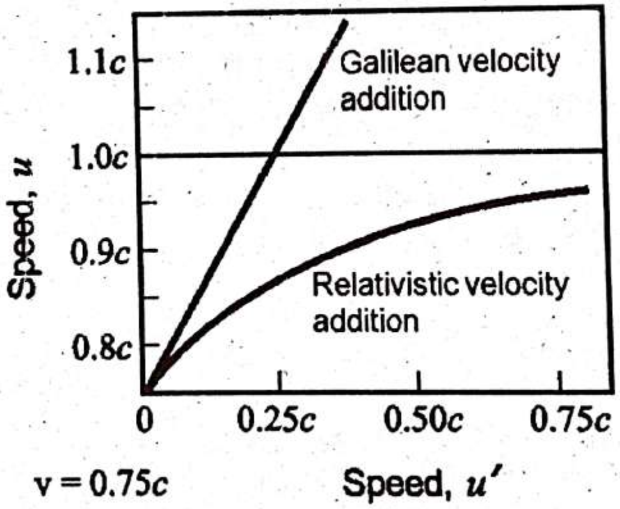


Figure 2.2: *Comparison of Classical Velocity and Relativistic Velocity*

2.2 Acceleration Lorentz Transformation

Q. Adding and subtracting the same term Derive relation for acceleration Lorentz transformations.

In Newtonian mechanics, acceleration is the time derivative of the velocity but in special relativity time and velocity are relative quantities. This requires a modified definition of acceleration.

In space, acceleration can be related with proper acceleration using Lorentz transformations. As a result equation of motion (Newton's second law) will also be modified.

In Newtonian mechanics, acceleration is given by

$$\mathbf{a} = \frac{d\mathbf{u}}{dt}$$

In relative motion, according to Galilean transformation

$$\mathbf{u}' = \mathbf{u} - \mathbf{v}, \qquad t' = t$$

Therefore acceleration is same in all inertial frames. i.e.,

$$\begin{aligned} \frac{d\mathbf{u}'}{dt'} &= \frac{d\mathbf{u}}{dt} \\ \mathbf{a}' &= \mathbf{a} \end{aligned}$$

Lorentz transformations for velocity and time are

$$\begin{aligned} u_x' &= \frac{dx'}{dt'} = \frac{u_x - v}{1 - \frac{v}{c^2}u_x} \\ u_y' &= \frac{dy'}{dt'} = \frac{u_y}{\gamma\left(1 - \frac{v}{c^2}u_x\right)} \\ u_z' &= \frac{dz'}{dt'} = \frac{u_z}{\gamma\left(1 - \frac{v}{c^2}u_x\right)} \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right) \end{aligned}$$

Taking differentials of above equations

$$\begin{aligned} du_x' &= \frac{\left(1 - \frac{v}{c^2}u_x\right)(du_x - 0) - (u_x - v)\left(0 - \frac{v}{c^2}du_x\right)}{\left(1 - \frac{v}{c^2}u_x\right)^2} \quad \because dv = 0 = \text{uniform velocity} \\ &= \frac{\left(1 - \frac{v}{c^2}u_x + \frac{v}{c^2}u_x - \frac{v^2}{c^2}\right)}{\left(1 - \frac{v}{c^2}u_x\right)^2}du_x \\ &= \frac{\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \frac{v}{c^2}u_x\right)^2}du_x \\ du_x' &= \frac{du_x}{\gamma^2\left(1 - \frac{v}{c^2}u_x\right)^2} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} du'_y &= \frac{\gamma\left(1-\frac{v}{c^2}u_x\right)du_y - u_y\gamma\left(0-\frac{v}{c^2}du_x\right)}{\gamma^2\left(1-\frac{v}{c^2}u_x\right)^2} \\ du'_y &= \frac{\gamma\left(1-\frac{v}{c^2}u_x\right)du_y + u_y\gamma\frac{v}{c^2}du_x}{\gamma^2\left(1-\frac{v}{c^2}u_x\right)^2} \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} du'_z &= \frac{\gamma\left(1-\frac{v}{c^2}u_x\right)du_z - u_z\gamma\left(0-\frac{v}{c^2}du_x\right)}{\gamma^2\left(1-\frac{v}{c^2}u_x\right)^2} \\ du'_z &= \frac{\gamma\left(1-\frac{v}{c^2}u_x\right)du_z + u_z\gamma\frac{v}{c^2}du_x}{\gamma^2\left(1-\frac{v}{c^2}u_x\right)^2} \end{aligned} \tag{2.6}$$

and

$$dt' = \gamma\left(dt-\frac{v}{c^2}dx\right) = \gamma\left(1-\frac{v}{c^2}\frac{dx}{dt}\right)dt = \gamma\left(1-\frac{v}{c^2}u_x\right)dt \tag{2.7}$$

Now dividing Eq.(2.4) by (2.7)

$$\begin{aligned} \frac{du'_x}{dt'} &= \frac{du_x}{\gamma^2\left(1-\frac{v}{c^2}u_x\right)^2}\cdot\frac{1}{\gamma\left(1-\frac{v}{c^2}u_x\right)dt} \\ a'_x &= \frac{a_x}{\gamma^3\left(1-\frac{v}{c^2}u_x\right)^3} \end{aligned} \tag{2.8}$$

Similarly dividing by Eq.(2.5) by (2.7)

$$\begin{aligned} \frac{du'_y}{dt'} &= \frac{\gamma\left(1-\frac{v}{c^2}u_x\right)du_y + u_y\gamma\frac{v}{c^2}du_x}{\gamma^2\left(1-\frac{v}{c^2}u_x\right)^2}\cdot\frac{1}{\gamma\left(1-\frac{v}{c^2}u_x\right)dt} \\ &= \frac{\gamma\left(1-\frac{v}{c^2}u_x\right)du_y}{\gamma^3\left(1-\frac{v}{c^2}u_x\right)^3dt} + \frac{u_y\gamma\frac{v}{c^2}du_x}{\gamma^3\left(1-\frac{v}{c^2}u_x\right)^3dt} \\ a'_y &= \frac{a_y}{\gamma^2\left(1-\frac{v}{c^2}u_x\right)^2} + \frac{\frac{u_yv}{c^2}a_x}{\gamma^2\left(1-\frac{v}{c^2}u_x\right)^3} \end{aligned} \tag{2.9}$$

and by symmetry

$$a'_z = \frac{a_z}{\gamma^2\left(1-\frac{v}{c^2}u_x\right)^2} + \frac{\frac{u_z v}{c^2}a_x}{\gamma^2\left(1-\frac{v}{c^2}u_x\right)^3} \tag{2.10}$$

Equations (2.8), (2.9) and (2.10) are called **Lorentz transformations for acceleration.**

2.3 Momentum Lorentz Transformation

Q. Derive relation for momentum Lorentz transformations.

In Newtonian mechanics, *momentum* is *the product of mass and velocity.* But in special relativity the definition of momentum will be modified due to Lorentz transformation of space and time coordinates.

Momentum is given by

$$\mathbf{p} = m\mathbf{v} = m\frac{d\mathbf{r}}{dt}$$

The momentum $\mathbf{p}'$ in moving frame S' is given by

$$\mathbf{p}' = m\mathbf{v}' = m\frac{d\mathbf{r}'}{dt}$$

Coordinates Lorentz transformation is given by

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

So x-component of momentum $\mathbf{p}'$ will be

$$p'_x = m\frac{dx'}{dt} = m\gamma\frac{d}{dt}(x - vt) = m\gamma(u_x - v) = \gamma(mu_x - mv) = \gamma\left(p_x - \frac{E}{c^2}v\right) \tag{2.11}$$

and the y-component of momentum will be

$$p'_y = m\frac{dy'}{dt} = m\frac{dy}{dt} = mu_y = p_y \tag{2.12}$$

and by symmetry

$$p'_z = p_z \tag{2.13}$$

Equations (2.11), (2.12) and (2.13) are called **Lorentz transformations for momentum.**

2.4 Force Lorentz Transformation

Q. Derive relation for force Lorentz transformations.

In Newtonian mechanics, force is the time rate of change of momentum but in special relativity time and momentum are not absolute but transform under Lorentz transformations. Thus force in special relativity will not have simple form.

The Lorentz transformations for momentum and time for motion along x-axis are

$$\begin{aligned} p'_x &= \gamma\left(p_x - \frac{Ev}{c^2}\right) && (2.14)\\ p'_y &= p_y && (2.15)\\ p'_z &= p_z && (2.16)\\ t' &= \gamma\left(t - \frac{v}{c^2}x\right) && (2.17) \end{aligned}$$

To find the force, resolve force in two types of components:

1. Parallel Components ($F_x = F_{\|}$)
2. Perpendicular Components ($F_{y,z} = F_{\perp}$)

2.4.1 Parallel Components ($F_x = F_{\|}$)

According to Newton's second law the force in moving frame S' will be

$$F'_x = \frac{dp'_x}{dt'} = \frac{dp'_x/dt}{dt'/dt}$$

Using Eq.(2.14) and (2.17)

$$\begin{aligned} F'_x &= \frac{\gamma\left(\frac{dp_x}{dt} - \frac{v}{c^2}\frac{dE}{dt}\right)}{\gamma\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)}\\ F'_x &= \frac{F_x - \frac{v}{c^2}\frac{dE}{dt}}{1 - \frac{vu_x}{c^2}} && (2.18) \end{aligned}$$

Now, we calculate $\frac{dE}{dt}$.

Energy is related with momentum as

$$\begin{aligned} E^2 &= c^2p^2 + E_o^2\\ E^2 &= c^2(\mathbf{p}\cdot\mathbf{p}) + E_o^2 \end{aligned}$$

Differentiating with respect to time

$$\begin{aligned} 2E\frac{dE}{dt} &= c^2\left(\mathbf{p}\cdot\frac{d\mathbf{p}}{dt} + \frac{d\mathbf{p}}{dt}\cdot\mathbf{p}\right) + 0 \quad \because \frac{dE}{dt} = 0\\ 2E\frac{dE}{dt} &= 2c^2\mathbf{p}\cdot\frac{d\mathbf{p}}{dt} \end{aligned}$$

$$\begin{aligned}\frac{dE}{dt} &= \frac{c^2\mathbf{p}}{E}\cdot\mathbf{F}\\ &= \frac{\mathbf{p}}{m}\cdot\mathbf{F} \quad \because E = mc^2\\ \frac{dE}{dt} &= \mathbf{u}\cdot\mathbf{F}\end{aligned}$$

Putting in Eq.(2.18)

$$F'_x = \frac{F_x - \frac{v}{c^2}(\mathbf{u}\cdot\mathbf{F})}{1 - \frac{vu_x}{c^2}}$$

Above equation gives the required parallel component of force.

2.4.2 Perpendicular Components ($F_{x,y} = F_\perp$)

Now perpendicular components (F_y and F_z) of the force are

$$F'_y = \frac{dp'_y}{dt'} = \frac{dp'_y/dt}{dt'/dt}$$

Using Eq.(2.15) and (2.17)

$$\begin{aligned}F'_y &= \frac{dp_y/dt}{\gamma\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)}\\ F'_y &= \frac{F_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}\end{aligned} \tag{2.19}$$

and by symmetry z-component of the force from Eq.(2.16) is

$$F'_z = \frac{F_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} \tag{2.20}$$

Equations (2.19) and (2.20) are the perpendicular components of the force.

2.5 Relativistic Doppler Effect

Q. Derive relation for relativistic Doppler's effect. Also describe its types and cases.

1. Longitudinal Doppler Effect
2. Transverse Doppler Effect

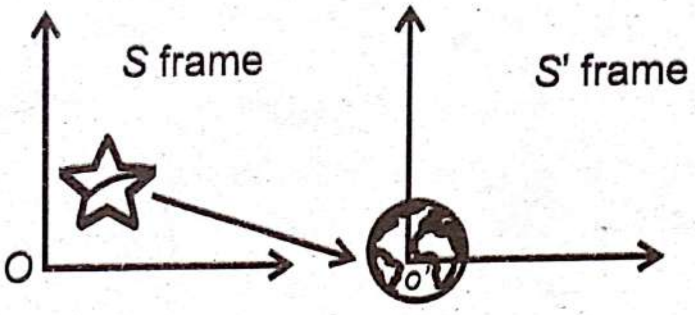


Figure 2.3: *The Relativistic Doppler Effect.*

The change in the frequency of light due to relative motion of the source and observer is called relativistic Doppler Effect.

Consider two inertial frames of reference S and S'. The frame S' is moving along x-axis with uniform velocity v relative to the frame S. Then according to inverse Lorentz transformation equations

$$\begin{aligned} x &= \gamma\left(x' + vt\right) \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' + \frac{v}{c^2}x\right) \\ u_x &= \frac{v + u'_x}{1 + \frac{v}{c^2}u'_x} \end{aligned}$$

	Frame S'	Frame S
Event 1	$x'_1 = 0$	$x_1 = 0$
Light Emission	$t'_1 = 0$	$t_1 = 0$
Event 2	$x'_2 = 0$	$x_2 = \gamma\frac{v}{f'}$
Light Reception	$t'_2 = \frac{1}{f'}$	$t_2 = \gamma\frac{1}{f'}$

Suppose two light pulses are emitted from the Star towards the Earth at time moments t'_1 and t'_2, and are observed in frames S and S'. In frame S' light pulse is sent at $t'_1 = 0$ and received at $t'_2 = \frac{1}{f'}$ where f' is the frequency of light pulse (Fig:2.3). In frame S light pulse is sent at $t_1 = 0, x_1 = 0$ and received at $t_2 = \gamma\frac{1}{f'}$ and $x_2 = \gamma\frac{v}{f'}$ using Lorentz transformations. The time taken by light pulse in frame S will be

$$\begin{aligned} \tau &= t_2 - t_1 - \frac{x_2}{c} \\ &= \frac{\gamma}{f'} - 0 - \frac{\gamma v}{f'c} \\ \tau &= \frac{\gamma}{f'}\left(1 - \frac{v}{c}\right) \end{aligned}$$

Replacing τ with $\frac{1}{f}$

$$\begin{aligned}
\frac{1}{f} &= \frac{\gamma}{f'}\left(1-\frac{v}{c}\right) \\
f' &= \gamma\left(1-\frac{v}{c}\right) f \\
&= \frac{\left(1-\frac{v}{c}\right)}{\sqrt{1-\frac{v^2}{c^2}}} f \\
&= \frac{\left(1-\frac{v}{c}\right)}{\sqrt{\left(1-\frac{v}{c}\right)\left(1+\frac{v}{c}\right)}} f \\
&= \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} f \\
f' &= \sqrt{\frac{c-v}{c+v}} f
\end{aligned}$$

This shows that when the source moves toward the observer, the observed frequency f' is greater than the emitted frequency f. The difference $f - f' = \Delta f$ is called the Doppler frequency shift. When $\frac{v}{c}$ is much smaller than 1, the fractional shift $\frac{\Delta f}{f}$ is also small and is approximately equal to $\frac{v}{c}$:

$$\frac{\Delta f}{f} = \frac{v}{c}$$

Until early in the 20th century it was assumed that the universe was static; stars might move relative to each other. But measurements, in 1912, indicated that the universe is not static. The motions of galaxies relative to the earth can be measured by observing the shifts in the wavelengths of their spectra. For distant galaxies these shifts are always toward longer wavelength, so they appear to be receding from us and from each other.

In terms of wavelength $f = \frac{c}{\lambda}$ and $f' = \frac{c}{\lambda'}$, so

$$\begin{aligned}
\frac{c}{\lambda'} &= \sqrt{\frac{c-v}{c+v}}\frac{c}{\lambda} \\
\frac{1}{\lambda'} &= \sqrt{\frac{c-v}{c+v}}\frac{1}{\lambda} \\
\frac{\lambda}{\lambda'} &= \frac{\sqrt{c-v}}{\sqrt{c+v}} \\
\frac{\lambda'}{\lambda} &= \frac{\sqrt{c+v}}{\sqrt{c-v}}
\end{aligned}$$

Using the radial Doppler Effect for a moving source above relation relates wavelength in the frame of source and wavelength in the fame of the observer,λ_o is the wave length in the frame of source and λ is the wave length in the frame of the observer.

Wavelengths from receding sources are always shifted toward longer wavelengths; this increase in λ is called the **redshift** and wavelength from approaching sources are always shifted toward shorter wavelengths; this decrease in λ is called **blueshift.**

Energy of the source and observer are given by

$$E = hf \qquad E' = hf'$$

Dividing these two equations

$$\begin{aligned} \frac{E'}{E} &= \frac{hf'}{hf} \\ \frac{E'}{E} &= \frac{f'}{f} \end{aligned}$$

CASES:-

2.5.1 Source and Observer are Receding

If source and observer are approaching each other then velocity v will be positive. So the relation between f' and f will be

$$f' = \sqrt{\frac{c-v}{c+v}}f$$

2.5.2 Source and Observer are Approaching

If source and observer are approaching each other then velocity v will be negative. So the relation between f' and f will be

$$\begin{aligned} f' &= \sqrt{\frac{c-(-v)}{c+(-v)}}f \\ f' &= \sqrt{\frac{c+v}{c-v}}f \end{aligned}$$

2.5.3 Source and Observer are in Perpendicular Direction

The transverse Doppler Effect occurs when the source and observer are at the point of closest approach. Light received when the emitter is at closest approach is blueshifted relative to its source frequency. Light from when the source is seen at closest approach will be redshifted at the observer. This is called transverse Doppler Effect which is given by the formula

$$f' = \frac{f}{\sqrt{1-\frac{v^2}{c^2}}}$$

2.6 Aberration of Light

Q. Derive relation for abberation of Light.

> The phenomenon which produces an apparent motion of celestial objects about their true positions dependent on the velocity of the observer is called aberration of light.

Bradley discovered the phenomenon of aberration of light in 1927.

Suppose the velocity of the particles in frame S at a given time in xy-plane is $u = (u_x, u_y, 0)$ and in frame S' velocity is $u' = (u'_x, u'_y, 0)$(Fig:2.4). Then these velocities in component form will be

$$\begin{aligned} u_x &= u\cos\theta, & u_y &= u\sin\theta \\ u'_x &= u'\cos\theta', & u'_y &= u'\sin\theta' \end{aligned} \tag{2.21}$$

From Lorentz velocity transformation

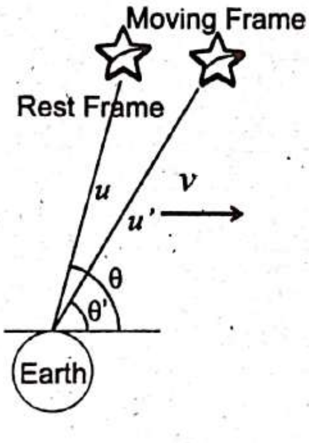


Figure 2.4: *Abberation of Light*

$$u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}}, \qquad u_y = \frac{u'_y}{\gamma\left(1 + \frac{vu'_x}{c^2}\right)}$$

Dividing these two equations

$$\frac{u_y}{u_x} = \frac{\dfrac{u'_y}{\gamma\left(1 + \frac{vu'_x}{c^2}\right)}}{\dfrac{u'_x + v}{1 + \frac{vu'_x}{c^2}}}$$

$$\frac{u_y}{u_x} = \frac{u'_y}{\gamma(u'_x + v)}$$

Using Eq.(2.21)

$$\frac{u\sin\theta}{u\cos\theta}=\frac{u'\sin\theta'}{\gamma(u'\cos\theta'+v)}$$

For light put $u=u'=c$

$$\tan\theta=\frac{c\sin\theta'}{\gamma(c\cos\theta'+v)}$$

To find $\sin\theta$ and $\cos\theta$ using trigonometric formulae.

$$\begin{aligned}
\csc^2\theta &= 1+\tan^2\theta\\
&= 1+\frac{c^2}{\sin^2\theta'}\gamma(c\cos\theta'+v)^2\\
\csc^2\theta &= 1+\frac{c^2}{\sin^2\theta'}\left(\frac{c^2}{c^2-v^2}\right)(c\cos\theta'+v)^2\\
&= 1+\frac{(c^2-v^2)\sin^2\theta'}{(c^2\cos^2\theta'+v^2+2cv\cos\theta')}\\
&= \frac{c^2\cos^2\theta'+v^2+2cv\cos\theta'+c^2\sin^2\theta'-v^2\sin^2\theta'}{(c\cos\theta'+v)^2}\\
&= \frac{c^2+v^2+2cv\cos\theta'+v^2\cos^2\theta'}{(c\cos\theta'+v)^2}\\
&= \frac{c^2+2cv\cos\theta'+v^2\cos^2\theta'}{(c\cos\theta'+v)^2}\\
&= \left(\frac{c+v\cos\theta'}{c\cos\theta'+v}\right)^2\\
\csc^2\theta &= \left(\frac{1+\frac{v}{c}\cos\theta'}{\cos\theta'+\frac{v}{c}}\right)^2\\
\csc\theta &= \frac{1+\frac{v}{c}\cos\theta'}{\cos\theta'+\frac{v}{c}}\\
\cos\theta=\frac{1}{\csc\theta} &= \frac{\cos\theta'+\frac{v}{c}}{1+\frac{v}{c}\cos\theta'}
\end{aligned}$$

Now to find $\sin\theta$ using

$$\begin{aligned}
\sin\theta &= \sqrt{1-\cos^2\theta}\\
&= \sqrt{1-\left(\frac{\cos\theta'+\frac{v}{c}}{1+\frac{v}{c}\cos\theta'}\right)^2}
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\frac{\left(1+\frac{v}{c}\cos\theta'\right)^2-\left(\cos\theta'+\frac{v}{c}\right)^2}{\left(1+\frac{v}{c}\cos\theta'\right)^2}}\\
&= \sqrt{\frac{1+\frac{v^2}{c^2}\cos^2\theta'+2\frac{v}{c}\cos\theta'-\cos^2\theta'-\frac{v^2}{c^2}-2\frac{v}{c}\cos\theta'}{\left(1+\frac{v}{c}\cos\theta'\right)^2}}\\
&= \sqrt{\frac{\sin^2\theta'-\frac{v^2}{c^2}(1-\cos^2\theta')}{\left(1+\frac{v}{c}\cos\theta'\right)^2}}\\
&= \sqrt{\frac{\sin^2\theta'-\frac{v^2}{c^2}\sin^2\theta'}{\left(1+\frac{v}{c}\cos\theta'\right)^2}}\\
\sin\theta &= \frac{\sqrt{1-\frac{v^2}{c^2}}\sin\theta'}{1+\frac{v}{c}\cos\theta'}
\end{aligned}
$$

Now take $\sin\theta-\sin\theta'$

$$
\begin{aligned}
\sin\theta-\sin\theta &= \frac{\sqrt{1-\frac{v^2}{c^2}}\sin\theta'}{1+\frac{v}{c}\cos\theta'}-\sin\theta'\\
&= \left(\frac{\sqrt{1-\frac{v^2}{c^2}}}{1+\frac{v}{c}\cos\theta'}-1\right)\sin\theta'\\
&= \left(\frac{\sqrt{1-\frac{v^2}{c^2}}-1-\frac{v}{c}\cos\theta'}{1+\frac{v}{c}\cos\theta'}\right)\sin\theta'\\
&\approx \left(1-\frac{v}{c}\cos\theta'-1\right)\sin\theta'\\
&= -\frac{v}{c}\sin\theta'\cos\theta'
\end{aligned}
$$

As $\sin\theta-\sin\theta'=2\sin\frac{1}{2}(\theta'-\theta)\cos\frac{1}{2}(\theta+\theta')\approx(\theta'-\theta)\cos\theta'$
where $\cos\frac{1}{2}(\theta+\theta')\approx\cos\theta'$, so

$$
(\theta'-\theta)\cos\theta' = -\frac{v}{c}\sin\theta'\cos\theta'
$$

$$\Delta\theta = \theta' - \theta = \frac{v}{c}\sin\theta' \tag{2.22}$$

Eq.(2.22) gives the relation between the angles θ and θ'. This formula relates the position of a star to its observed position. The displacement being caused by the motion of earth relative to the sun and finiteness of the velocity of light. This equation has a great significance in astronomy. Observations of stellar spectra determine the rates at which starts are moving towards or away from us.

KEY TERMS

- **Galilean velocity addition rules:** Velocity addition rules used for adding relative velocities in classical mechanics are called Galilean velocity addition rules.
- **Relativistic Doppler Effect:** The change in the frequency of light due to relative motion of the source and observer is called relativistic Doppler Effect.
- **Redshift:** Wavelengths from receding sources are always shifted toward longer wavelengths; this increase in wavelength is called the redshift
- **Blueshift:** wavelength from approaching sources are always shifted toward shorter wavelengths; this decrease in wavelength is called blueshift
- **Abberation of Light:** The phenomenon which produces an apparent motion of celestial objects about their true positions dependent on the velocity of the observer is called aberration of light.

KEY FORMULAE

Galilean velocity addition rules

$$\begin{aligned} u'_x &= u - v \\ u'_y &= u_y \\ u'_z &= u_z \end{aligned}$$

Lorentz velocity transformation

$$\begin{aligned} u'_x &= \frac{dx'}{dt'} = \frac{\gamma(dx - vdt)}{\gamma\left(dt - \frac{v}{c^2}dx\right)} = \frac{dt\left(\frac{dx}{dt} - v\right)}{dt\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)} = \frac{u_x - v}{1 - \frac{v}{c^2}u_x} \\ u'_y &= \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - \frac{v}{c^2}dx\right)} = \frac{dy}{\gamma dt\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)} = \frac{u_y}{\gamma\left(1 - \frac{v}{c^2}u_x\right)} \\ u'_z &= \frac{dz'}{dt'} = \frac{dz}{\gamma\left(dt - \frac{v}{c^2}dx\right)} = \frac{dz}{\gamma dt\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)} = \frac{u_z}{\gamma\left(1 - \frac{v}{c^2}u_x\right)} \end{aligned}$$

Inverse Lorentz velocity transformation

$$u_x = \frac{u'_x + v}{1 + \frac{v}{c^2}u'_x}$$

$$\begin{aligned} u_y &= \frac{u'_y}{1+\frac{v}{c^2}u'_x} \\ u_z &= \frac{u'_z}{1+\frac{v}{c^2}u'_x} \end{aligned}$$

Acceleration Lorentz transformation

$$\begin{aligned} a'_x &= \frac{a_x}{\gamma^3\left(1-\frac{v}{c^2}u_x\right)^3} \\ a'_y &= \frac{a_y}{\gamma^2\left(1-\frac{v}{c^2}u_x\right)^2}+\frac{\frac{u_y v}{c^2}a_x}{\gamma^2\left(1-\frac{v}{c^2}u_x\right)^3} \end{aligned}$$

$$a'_z = \frac{a_z}{\gamma^2\left(1-\frac{v}{c^2}u_x\right)^2}+\frac{\frac{u_z v}{c^2}a_x}{\gamma^2\left(1-\frac{v}{c^2}u_x\right)^3}$$

Momentum Lorentz transformation

$$\begin{aligned} p'_x &= \gamma\left(p_x-\frac{E}{c^2}v\right) \\ p'_y &= p_y \end{aligned}$$

$$p'_z = p_z$$

Force Lorentz transformation

$$\begin{aligned} F'_x &= \frac{F_x-\frac{v}{c^2}(\mathbf{u}\cdot\mathbf{F})}{1-\frac{vu_x}{c^2}} \\ F'_y &= \frac{dp'_y}{dt'}=\frac{dp'_y/dt}{dt'/dt} \\ F'_z &= \frac{F_z}{\gamma\left(1-\frac{vu_x}{c^2}\right)} \end{aligned}$$

Relativistic Doppler effect (frequency form)

$$f' = \sqrt{\frac{c-v}{c+v}}f$$

Relativistic Doppler effect (wavelength form)

$$\frac{\lambda'}{\lambda} = \frac{\sqrt{c+v}}{\sqrt{c-v}}$$

Relativistic Doppler effect (Source and Observer are receding)

$$f' = \sqrt{\frac{c-v}{c+v}}f$$

Relativistic Doppler effect (Source and Observer are approaching)

$$f' = \sqrt{\frac{c+v}{c-v}}f$$

Relativistic Doppler effect (Source and observer are in perpendicular direction)

$$f' = \frac{f}{\sqrt{1-\frac{v^2}{c^2}}}$$

Aberration of light

$$\Delta\theta = \theta' - \theta = \frac{v}{c}\sin\theta'$$

EXERCISE

Problems

1. An electron of mass $m = 9.11 \times 10^{-31} kg$ has speed $4c/5$. What is the magnitude of its (relativistic) momentum?

2. Compute the kinetic energy of a muon (mass $m_\mu = 1.88 \times 10^{-28} kg$) travelling with speed $9c/10$.

3. The proton has mass $m_p = 1.67 \times 10^{-27} kg$. Compute the total (relativistic) energy of a proton moving with speed $v = 3c/5$.

4. At what speed is the total energy of a particle twice the mass energy?

5. In a nuclear fission of uranium-235 caused by the absorption of a neutron, nuclei of krypton and barium are produced, and three neutrons are emitted. The difference in the total mass of the particles present at the start of the process and those present at the end is $3.08 \times 10^{-28} kg$. What is the energy released in this process, in both joules and electronvolts ($1eV = 1.60 \times 10^{-19} J$)?

6. Given that the binding energy of a hydrogen atom is 13.6 eV, what is the difference between the mass of a hydrogen atom and the masses of its constituent electron and proton?

7. In frame of reference S, an electron moving along the x-axis has energy $3m_ec^2$ and momentum magnitude $\sqrt{8}m_ec$. Use the transformations of energy and momentum to find the energy and momentum magnitude observed in frame S' moving with speed $4c/5$ relative to S in the positive x-direction.

8. A car deriver accused of cross on a red light, claims that it appeared to be green because of the Doppler effect. If this was true, what was the speed of the car? (Note: the relevant ratio of frequency may be takes as $\frac{\nu\,(green)}{\nu\,(red)} = 2.1$

9. A particle has velocity $u = (u_1, u_2, u_3)$ in S and $u' = (u'_1, u'_2, u'_3)$ in where is moving along the x direction with uniform velocity v. Using velocity transformation formulae show that

$$\left(c^2 - u^2\right) = c^2 \frac{\left(c^2 - u'^2\right)\left(c^2 - v^2\right)}{\left(c^2 + vu_1'\right)}$$

Deduce that if the speed of particle is less than the speed of the light c in any one inertial frame then it is less than c in every inertial frame.

10. An excited nucleus at red with rest mass M_o emits a photon in a transition to the ground state and loses energy ΔE in the process; the nucleus also undergoes a recoil in the process. Show that the frequency ν of the photon is given by

$$\nu = \frac{e}{h}\left[1 - \frac{e}{2M_oc^2}\right]$$

11. Show that the frequency ν of light emitted from a moving source is given by

$$\nu = \frac{\nu_o}{\gamma\,(1 - \beta\cos\theta)}$$

Where ν_o is the proper frequency of light $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}, \beta = \dfrac{v}{c}$, ν being the speed of source and θ is the angle between the direction of motion and the emitted light signal.

Chapter 3

Energy-Mass Equivalence

Learning Objectives

In this chapter you will learn

- To derive relation for mass variation using Lorentz velocity transformations.
- To derive relation for relativistic mass
- To derive relation for relativistic momentum
- To derive relation for relativistic force
- To derive mass-energy equivalence relation and its applications.
- To derive energy-momentum relation
- To derive a relation for mass-energy conservation
- About zero rest mass particles
- About Cherenkov effect and Cherenkov radiation
- Invariance of physical quantities

KEY TERMS

mass variation, relativistic mass, relativistic momentum, relativistic force, mass-energy equivalence, photons, neutrinos, tachyons, Cherenkov radiation, Cherenkov effect, invariance of physical quantities

3.1 Mass Variation using Lorentz velocity transformations

Q. Derive relations for mass variation using Lorentz velocity transformations.
Consider an elastic head-on collision between two similar point particles. Denote S' as the centre of mass frame in which the collision is symmetric, i.e., they appear with equal velocities and after the collision they move back with velocities $-u'$ and $+u'$ of the same magnitude. Let v is the velocity of S' with respect to S and u_1 and u_2 are velocities of particles in frame S (Fig:3.1).
Assuming the collision is perfectly elastic, the law of conservation of momentum is

$$m_1u_1 + m_2u_2 = m_1v + m_2v$$

As Inverse Lorentz transformation equations for velocity are

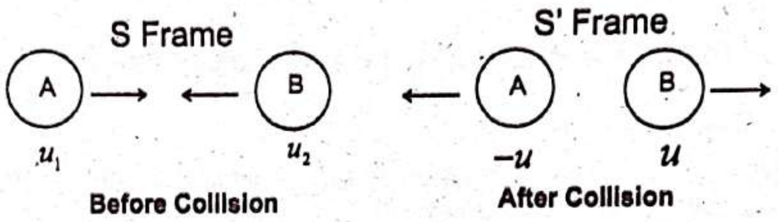


Figure 3.1: *Time in rest (left) and moving (right) frames is same*

$$u_1 = \frac{u' + v}{1 + \frac{u'v}{c^2}} \tag{3.1}$$

$$u_2 = \frac{-u' + v}{1 - \dfrac{u'v}{c^2}} \tag{3.2}$$

So the law of conservation of momentum equation becomes

$$\begin{aligned}
m_1\left[\frac{u' + v}{1 + \dfrac{u'v}{c^2}}\right] + m_2\left[\frac{-u' + v}{1 - \dfrac{u'v}{c^2}}\right] &= m_1v + m_2v \\
\frac{m_1c^2(u' + v)}{c^2 + u'v} + \frac{m_2c^2(-u' + v)}{c^2 - u'v} &= m_1v + m_2v \\
\frac{m_1c^2(u' + v)}{c^2 + u'v} - m_1v &= m_2v - \frac{m_2c^2(-u' + v)}{c^2 - u'v} \\
m_1\left[\frac{c^2(u' + v) - v(c^2 + u'v)}{c^2 + u'v}\right] &= m_2\left[\frac{v(c^2 - u'v) - c^2(-u' + v)}{c^2 - u'v}\right] \\
m_1\left[\frac{c^2 - v^2}{c^2 + u'v}\right]u' &= m_2\left[\frac{c^2 - v^2}{c^2 - u'v}\right]u' \\
\frac{m_1}{m_2} &= \frac{c^2 + u'v}{c^2 - u'v}
\end{aligned}$$

$$\frac{m_1}{m_2} = \frac{1+\dfrac{u'v}{c^2}}{1-\dfrac{u'v}{c^2}} \tag{3.3}$$

From Eq.(3.1) and (3.2), we calculate the relation for the right hand side of Eq.(3.3)

$$1-\frac{u_1^2}{c^2} = 1-\frac{(u'+v)^2}{c^2\left(1+\dfrac{u'v}{c^2}\right)^2}, \qquad 1-\frac{u_2^2}{c^2} = 1-\frac{(-u'+v)^2}{c^2\left(1-\dfrac{u'v}{c^2}\right)^2}$$

$$= 1-\frac{c^2(u'+v)^2}{(c^2+u'v)^2}, \qquad = 1-\frac{c^2(-u'+v)^2}{(c^2-u'v)^2}$$

$$= \frac{(c^2+u'v)^2-c^2(u'+v)^2}{(c^2+u'v)^2}, \qquad = \frac{(c^2-u'v)^2-c^2(-u'+v)^2}{(c^2-u'v)^2}$$

$$1-\frac{u_1^2}{c^2} = \frac{c^4+u'^2v^2-c^2u'^2-c^2v^2}{(c^2+u'v)^2}, \qquad 1-\frac{u_2^2}{c^2} = \frac{c^4+u'^2v^2-c^2u'^2-c^2v^2}{(c^2-u'v)^2}$$

Dividing these equations mutually

$$\frac{1-\dfrac{u_2^2}{c^2}}{1-\dfrac{u_1^2}{c^2}} = \frac{(c^2+u'v)^2}{(c^2-u'v)^2}$$

$$\sqrt{\frac{1-\dfrac{u_2^2}{c^2}}{1-\dfrac{u_1^2}{c^2}}} = \frac{1+\dfrac{u'v}{c^2}}{1-\dfrac{u'v}{c^2}}$$

Comparing with Eq.(3.3)

$$\frac{m_1}{m_2} = \sqrt{\frac{1-\dfrac{u_2^2}{c^2}}{1-\dfrac{u_1^2}{c^2}}}$$

Let the second particle is at rest in frame S then putting $u_1 = v, m_1 = m, u_2 = 0$ and $m_2 = m_o$ in above equation

$$\frac{m}{m_o} = \frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}$$

$$m = \frac{m_o}{\sqrt{1-\dfrac{v^2}{c^2}}}$$

This equation represents the equation of the variation of mass with velocity.

3.2 Relativistic Mass

Q. Derive relations for relativistic mass.

The mass of an object which changes with speed of the object as its speed approaches to the speed of light is called relativistic mass.

Relativistic mass increases with velocity and tend to infinity when the velocity approaches the speed of light. Relativistic mass becomes equal to rest mass if an object is at rest ($v = 0$). i.e.,

$$m_r = m_o$$

Relativistic mass is given by

$$m_r = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma m_o \tag{3.4}$$

Eq.(3.16) is the speed dependent mass. It may also be written in terms of total energy as

$$m_r = \frac{E}{c^2} \tag{3.5}$$

Eqns.(3.16) and (3.5) both represent the relativistic masses.

From rest mass energy

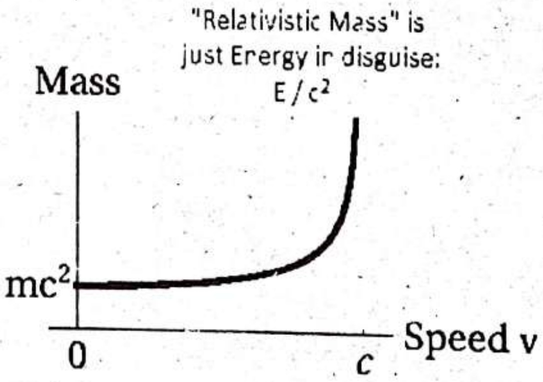


Figure 3.2: *Relativistic Mass as a function of speed*

$$m_o = \frac{E_o}{c^2}$$

In 1934, Tolman argued that the relativistic mass formula $m_r = \frac{E}{c^2}$ is applicable to all types of particles but the formula $m_r = \gamma m_o$ is only applicable to particles moving slower than speed of light. He also remarked the experimental verification of the relation in case of moving electron.

The rest mass of an electron is $9.1 \times 10^{-31} kg$ and it moves with the speed of $4.5 \times 10^5 ms^{-1}$ then its relativistic mass will be

$$m_r = \frac{9.1 \times 10^{-31}}{\sqrt{1 - \left(\frac{4.5 \times 10^5}{3 \times 10^8}\right)^2}} = 9.8 \times 10^{-31} kg$$

Similarly, the momentum in terms of relativistic mass becomes

$$p = m_r v$$

And Newton's second law becomes

$$F = \frac{dp}{dt} = \frac{d(m_r v)}{dt}$$

Note: The concept of relativistic mass was never used by Einstein but it was first used by the chemist G.N.Lewis in 1908. R.C.Tolman derived the formula for speed-dependent mass. In 1948 Einstein cautioned that it is not proper to speak of the mass $m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}}$ of a moving object because no clear definition can be given for m. It is better to restrict oneself to the rest mass m_o. Einstein never approved of relativistic mass by Eugene Hecht.

3.3 Relativistic Momentum

Q. Derive relations for relativistic momentum.

The concept of momentum was well introduced in Newtonian mechanics but it is limited to slow moving object. The quantity must be reformulated for high speed moving objects. The relativistic momentum relation required that **p** must be conserved in a collision for all observers in relative motion at constant velocity. Also $\mathbf{p} = m\mathbf{v}$ holds in classical mechanics for $v \ll c$.

Consider a particle of mass m is moving with uniform velocity **v** and coordinate intervals are $(\Delta x, \Delta y, \Delta z)$. Then particle's velocity components will be

$$\mathbf{v} = (v_x, v_y, v_z) = \left(\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}, \frac{\Delta z}{\Delta t}\right) \tag{3.6}$$

Components Δx, Δy, Δz and Δt transform according to Lorentz transformation in a complicated way but momentum would transform in a simple way, i.e., Δt to be replaced by proper time interval $\Delta\tau$ which is invariant. Thus relativistic momentum of the particle would be

$$\mathbf{p} = m\left(\frac{\Delta x}{\Delta\tau}, \frac{\Delta y}{\Delta\tau}, \frac{\Delta z}{\Delta\tau}\right)$$

As $\Delta\tau = \frac{\Delta t}{\gamma}$, so

$$\mathbf{p} = m\gamma\left(\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}, \frac{\Delta z}{\Delta t}\right)$$

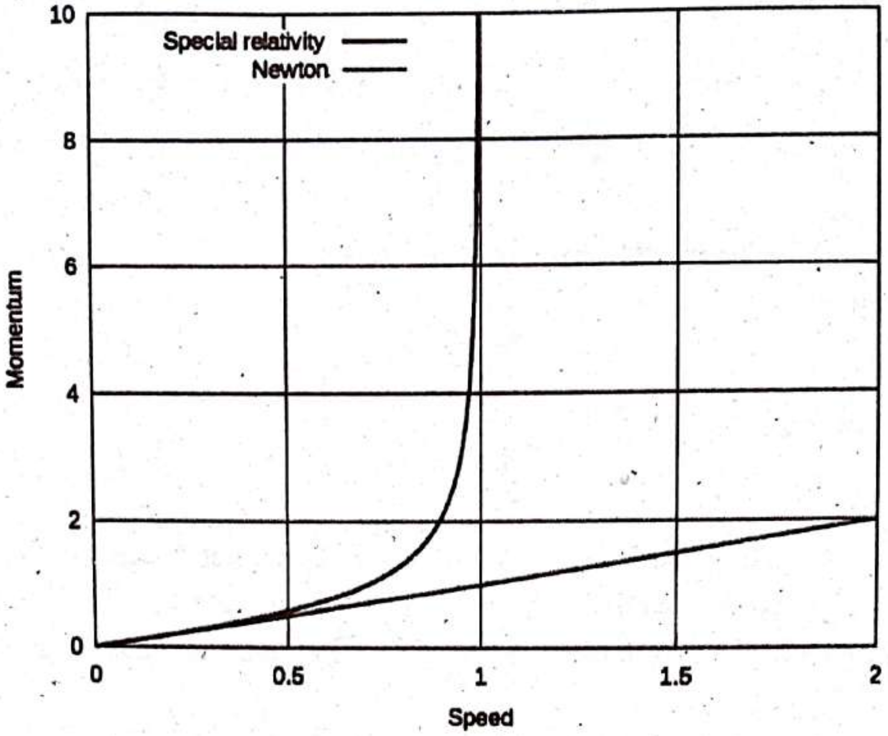


Figure 3.3: *Comparison of Newtonian Momentum and Relativistic Momentum*

Using Eq.(3.16)

$$\mathbf{p} = m\gamma\mathbf{v}$$
$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}} \quad \left(\because \gamma = \sqrt{1-\frac{v^2}{c^2}}\right)$$

This is an expression for relativistic momentum and its value approaches to infinity as speed approaches to speed of light (Fig.3.3).

There are some issues to be discussed here:

- *Classical momentum is limiting case of relativistic momentum.*
  The relativistic momentum $\mathbf{p} = m\gamma\mathbf{v}$ differs from classical momentum only by a Lorentz factor γ. For small speeds $\gamma = \sqrt{1-\frac{v^2}{c^2}} \approx 1$ then $\mathbf{p} = m\mathbf{v}$ which is equal to classical momentum.
- *Conservation of momentum in inertial frame.*

Consider two particles collision in reference frame S then

$$m_1\gamma u_1 + m_2\gamma u_2 = m_1\gamma v_1 + m_2\gamma v_2$$

While for collision in reference frame S'

$$m_1\gamma u_1' + m_2\gamma u_2' = m_1\gamma v_1' + m_2\gamma v_2'$$

Masses are not primed quantities because they are invariant.

Let initial momenta in reference frame S be $\mathbf{p}_1$ and $\mathbf{p}_2$ and final momenta be $\bar{\mathbf{p}_1}$ and $\bar{\mathbf{p}_2}$ then relativistic momentum conservation implies that

$$\mathbf{p}_1 + \mathbf{p}_2 = \bar{\mathbf{p}}_1 + \bar{\mathbf{p}}_2$$

Under a Lorentz transformation to other reference frame S' momenta will be $\mathbf{p}_1'$ and $\mathbf{p}_2'$ before collision and $\bar{\mathbf{p}}_1'$ and $\bar{\mathbf{p}}_2'$ after collision. So

$$\mathbf{p}_1' + \mathbf{p}_2' = \bar{\mathbf{p}}_1' + \bar{\mathbf{p}}_2'$$

If relativistic momentum is conserved in one inertial frame then it will be conserved in all inertial frames.

3.4 Relativistic Force

Q. Derive relations for relativistic force.

Time rate of change of relativistic momentum is called relativistic force.

In Newtonian mechanics, Newton's second law is in the form

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

In relativity, momentum is

$$\mathbf{p} = m\gamma\mathbf{v}$$

so above equation becomes

$$\begin{aligned}
\mathbf{F} &= \frac{d(\gamma m\mathbf{v})}{dt} \\
&= m\frac{d}{dt}\left[\left(1-\frac{v^2}{c^2}\right)^{-1/2}\right]\mathbf{v} \\
\mathbf{F} &= m\left(1-\frac{v^2}{c^2}\right)^{-1/2}\frac{d\mathbf{v}}{dt} - \frac{1}{2}m\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left(-\frac{2\mathbf{v}}{c^2}\cdot\frac{d\mathbf{v}}{dt}\right)\mathbf{v} \\
\mathbf{F} &= m\gamma\mathbf{a} + m\gamma^3\left(\frac{\mathbf{v}}{c^2}\cdot\mathbf{a}\right)\mathbf{v} \qquad (3.7)
\end{aligned}$$

Eq.(3.7) is the relativistically correct form of Newton's Second Law and are also called relativistic force equation. $m\gamma$ is called transverse mass and $m\gamma^3$ as longitudinal mass. Hence, at high speeds the force and acceleration need not to have the same direction. This equation involves two components, first perpendicular to v and second parallel to v.

For a particle moving parallel to x-direction, i.e., $\mathbf{v} = (v_x, 0, 0)$ and $\mathbf{a} = (a_x, 0, 0)$ the component of force along $x-$axis Eq.(3.7) will be

$$F_x = m\gamma a_x + \frac{m\gamma^3}{c^2}v_x^2 a_x \qquad \because \mathbf{v}\cdot\mathbf{a} = v_x a_x$$

$$
\begin{aligned}
&= m\gamma a_x\left(1+\gamma^2\frac{v^2}{c^2}\right) \qquad \because v_x^2 = v^2 \quad \text{for} \quad x-\text{direction}\\
&= m\gamma a_x\left[1+\frac{v^2/c^2}{\left(1-\frac{v^2}{c^2}\right)}\right]\\
&= \frac{ma_x}{\sqrt{1-\frac{v^2}{c^2}}}\frac{1}{\left(1-\frac{v^2}{c^2}\right)}\\
&= \frac{ma_x}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}\\
F_x &= m\gamma^3 a_x \qquad (3.8)
\end{aligned}
$$

In Eq.(3.8), F_x and a_x are parallel to each other and both are parallel to v.

Now, again motion along $x-$axis i.e., $\mathbf{v} = (v_x, 0, 0)$ and $\mathbf{a} = (a_x, 0, 0)$ the component of force along $y-$axis, Eq.(3.7) will be

$$
\begin{aligned}
F_y &= m\gamma a_y + \frac{m\gamma^3}{c^2}(0)\mathbf{v}\\
&= m\gamma a_y = \frac{ma_y}{\sqrt{1-\frac{v^2}{c^2}}}\\
F_y &= m\gamma a_y \qquad (3.9)
\end{aligned}
$$

In Eq.(3.9), F_y and a_y are parallel but both are perpendicular to v_x.

Now, again motion along $x-$axis i.e., $\mathbf{v} = (v_x, 0, 0)$ and $\mathbf{a} = (a_x, 0, 0)$ the component of force along $z-$axis, Eq.(3.7) will be

$$
\begin{aligned}
F_z &= m\gamma a_z + \frac{m\gamma^3}{c^2}(0)\mathbf{v}\\
&= m\gamma a_z = \frac{ma_y}{\sqrt{1-\frac{v^2}{c^2}}}\\
F_z &= m\gamma a_z \qquad (3.10)
\end{aligned}
$$

In Eq.(3.10), F_z and a_z are parallel but both are perpendicular to v_x

3.5 Mass Energy Equivalence

Q. Derive relations for mass energy equivalence.

Albert Einstein's most famous formula is

$$E = mc^2$$

Mass-energy equivalence appeared in special relativity as a paradox described by Henri Poincaré. Einstein, in his paper "Does the inertia depend upon its energy-content?" first time, he, proposed the equivalence of mass and energy.

According to this relation anybody which is stationary still has some energy which is rest mass energy but when body is in motion then its total energy is greater than rest mass energy. Likewise its total mass also becomes greater than its rest mass.

Consider a particle of mass m is moving uniformly with velocity v then the kinetic energy of a particle is determined as

$$K.E. = W = \int_0^v F dx$$

where F is the force along x-axis and W is work done in accelerating particle from rest to final speed v.

According to Newton's second law

$$F = \frac{dp}{dt}$$

So

$$K.E. = \int_0^v \frac{dp}{dt} dx = \int_0^v \frac{dx}{dt} dp = \int_u^v v dp$$

As $p = mv$ then

$$\begin{aligned} K.E. &= \int_u^v vd(mv) \\ &= \int_u^v v\,(mdv + vdm) \\ K.E. &= \int_u^v (mvdv + v^2 dm) \end{aligned} \tag{3.11}$$

To define relativistic kinetic energy of a particle of mass m using relativistic mass relation

$$\begin{aligned} m &= \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}} \\ m^2 &= \frac{m_o^2}{\left(1-\frac{v^2}{c^2}\right)} \\ m^2\left(1-\frac{v^2}{c^2}\right) &= m_o^2 \\ m^2\left(\frac{c^2-v^2}{c^2}\right) &= m_o^2 \\ m^2c^2 - m^2v^2 &= m_o^2c^2 \end{aligned}$$

Taking differentials

$$\begin{aligned} (2mdm)c^2 - (2mdm)v^2 - m^2(2vdv) &= 0 \\ 2m(c^2dm - v^2dm - mvdv) &= 0 \end{aligned}$$

$$\begin{aligned} c^2dm - v^2dm - mvdv &= 0 \\ c^2dm &= mvdv + v^2dm \end{aligned}$$

Putting in Eq.(3.11)

$$K.E. = \int_0^v c^2dm$$

Transforming limits over mass

$$\begin{aligned} K.E. &= \int_{m_o}^{m} c^2dm \\ &= c^2\left|m\right|_{m_o}^{m} \\ &= (m - m_o)\,c^2 \\ &= mc^2 - m_oc^2 \\ K.E. &= E - E_o \end{aligned}$$

Where $E_o = m_oc^2$ is the rest mass energy of the particle and $E = mc^2$ is the relativistic energy of the particle in moving frame and this is also known as Einstein's mass energy equivalence formula. It is the best known result of special relativity also called **mass-energy equivalence.**

3.6 Relativistic Kinetic Energy

Q. Derive relations for relativistic kinetic energy.
Kinetic energy in relativistic form is

$$\begin{aligned} K.E. &= E - E_o \\ K.E. &= mc^2 - m_oc^2 \end{aligned}$$

Using relation for relativistic mass $m = \dfrac{m_o}{\sqrt{1 - \dfrac{v^2}{c^2}}}$ above relation can be written as

$$K.E. = \left(\frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} - m_o\right)c^2$$

$$K.E. = m_oc^2\left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1\right) \tag{3.12}$$

$$K.E. = m_oc^2(\gamma - 1) \tag{3.13}$$

This is the expression for the relativistic kinetic energy of a particle of mass m moving with speed v.

To analyse relativistic energy in classical limit Eq.(3.12) can be written as

$$
\begin{aligned}
K.E. &= m_o c^2 \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right) \\
K.E. &= m_o c^2 \left[\left(1 - \frac{v^2}{c^2} \right)^{-1/2} - 1 \right]
\end{aligned}
$$

Using binomial theorem expansion

$$
\begin{aligned}
K.E. &= m_o c^2 \left[1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\left(\frac{v^2}{c^2}\right)^2 + \ldots - 1 \right] \\
&= m_o c^2 \left[\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\left(\frac{v^2}{c^2}\right)^2 + \ldots \right] \\
K.E. &= \frac{1}{2}\frac{v^2}{c^2} m_o c^2 \left[1 + \frac{3}{4}\left(\frac{v^2}{c^2}\right) + \ldots \right] \\
K.E. &= \frac{1}{2} m_o v^2 \left[1 + \frac{3}{4}\left(\frac{v^2}{c^2}\right) + \ldots \right]
\end{aligned}
$$

For $v \ll c$ (classical limit)

$$
\begin{aligned}
K.E. &\cong \frac{1}{2} m_o v^2 (1 + 0 + \ldots) \\
K.E. &= \frac{1}{2} m_o v^2
\end{aligned}
$$

This is the expression for kinetic energy in Newtonian mechanics which confirms that the Newtonian mechanics is the limiting case of the relativistic mechanics.

3.7 Total Relativistic Energy

Q. Derive relations for total relativistic energy.

The kinetic energy of an object is defined to be the work done on the object in accelerating it from rest to speed v

$$KE = \int_0^v F dx$$

Using result for relativistic force $F = m\gamma^3 a$ above equation becomes

$$KE = \int_0^v \gamma^3 m a dx$$

The variable of integration is x, yet the integrand is expressed in terms of a and v(v is hidden inside γ). To solve this problem, writing above equation as

$$\begin{aligned}
K.E. &= \int_0^v \gamma^3 m\left(\frac{dv}{dt}\right)dx \\
&= \int_0^v \gamma^3 m\left(\frac{dx}{dt}\right)dv \\
&= \int_0^v \gamma^3 mvdv \\
K.E. &= \int_0^v \frac{mvdv}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}
\end{aligned}$$

This integral can be done by a simple u-substitution, $u = 1 - \frac{v^2}{c^2}$, i.e.,

$$\begin{aligned}
du &= -\frac{2v}{c^2}dv \\
vdv &= -\frac{c^2}{2}du
\end{aligned}$$

Kinetic energy becomes

$$\begin{aligned}
K.E. &= -\int_0^v \frac{mc^2}{2}\frac{du}{u^{3/2}} \\
K.E. &= -\frac{mc^2}{2}\int_0^v u^{-3/2}du \\
&= -\frac{mc^2}{2}\left(\frac{u^{-1/2}}{-1/2}\right)_0^v \\
&= mc^2\left(\frac{1}{\sqrt{u}}\right)_0^v \\
&= mc^2\left[\frac{1}{1-\frac{v^2}{c^2}}\right]_0^v \\
&= mc^2\left[\frac{1}{1-\frac{v^2}{c^2}}-1\right] \\
K.E. &= \gamma mc^2 - mc^2 \\
\gamma mc^2 &= K.E. + mc^2 \\
E_{total} &= K.E. + E_{rest}
\end{aligned}$$

Einstein identified the term γmc^2 as the total energy of the particle which is the sum of the relativistic kinetic energy and a completely new form of energy, the rest energy (mc^2). Particles have rest energy just by virtue of having mass. In fact, mass is simply a form of energy.

It was introduced in his paper "Does the inertia of a body depend upon its energy content?"

published just after his first paper.

It is not necessary for relativistic kinetic energy and rest mass energy to be conserved separately but their sum (total relativistic energy) must be conserved. This relation plays a vital role in explaining many nuclear phenomenon and particle creation.

3.8 Mass Energy Equivalence Applications

Q. Describe mass-energy equivalence applications.

Einstein's mass energy equivalence relation has many applications in Nuclear physics. Some of the applications are described below.

1. Binding energy or mass defect
2. Nuclear radioactivity
3. Nuclear reactions
4. Fission
5. Pair production
6. Stellar energy (fusion in stars)

3.8.1 The Mass-Energy in Uranium Fission

The process of splitting large nuclei into small nuclei is called nuclear fission.

When large nuclei, such as uranium-235 split into small nuclei through fission reaction, energy is released. The energy released is in very large amount that there is a significant decrease in mass. This means that some of the mass is converted to energy. The amount of mass lost in the fission process is approximately equal to $210MeV$ of energy which is calculated below. One of the examples of products formed is given below:

$$\begin{aligned} n_0^1 + {}^{235}_{92}U &\rightarrow {}^{236}_{92}U \\ {}^{236}_{92}U &\rightarrow {}^{144}_{56}Ba + {}^{89}_{36}Kr + 3n_0^1 + 210MeV \end{aligned}$$

Fermi originally split the uranium nuclei in 1934. He believed that certain elements could be produced by bombarding uranium with neutrons. Although he expected the new nuclei to have larger atomic numbers than the original uranium, he found that the formed nuclei were radioisotopes of lighter elements. These results were also correctly interpreted by Lise Meitner and Otto Frisch.

The enormous energy that's released from this splitting comes from how hard the protons are repelling each other with the Coulomb force, barely held together by the strong force.

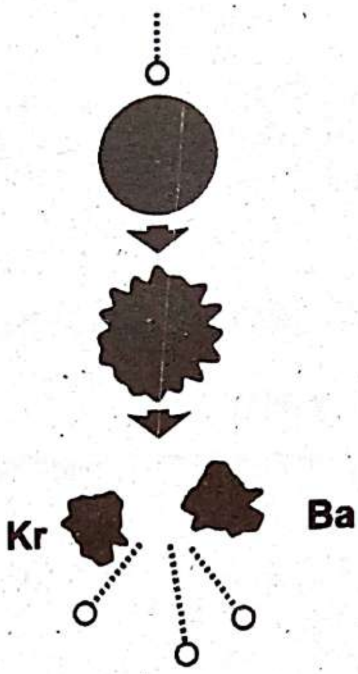


Figure 3.4: *Nuclear Fission Reaction*

In the uranium nucleus the nucleons are bound with an average energy of about $7.6MeV$ per nucleon. In the fission product nuclei, the medium binding energy per nucleon amounts to approximately $8.5MeV$. This difference in binding energy of $0.9MeV$ per nucleon is released in the nuclear fission. Since the uranium nucleus has 235 nucleons, a quantity of energy of about $210MeV$ is released per fission. It is made up of the following partial amounts:

- Kinetic energy of fission products $175MeV$,
- Kinetic energy of fission neutrons $5MeV$,
- Energy of the gamma radiation occurring during the fission $7MeV$,
- Energy of beta and gamma radiation during the decay of the radioactive fission products $13MeV$,
- Energy of the neutrinos $10MeV$

This huge amount of released energy is large enough to cause a measurable reduction in mass.

The total mass of each of the fission fragments is less than the mass of the starting nucleus. This missing mass is known as the mass defect and the amount of energy that binds the nuclei together is called binding energy.

All nuclei having this binding energy except hydrogen (which has just 1 proton and no neutrons). The binding energy available to each nucleon and this is called the binding energy per nucleon. This energy is required per nucleon to separate a nucleus. The products of fission are more stable so that the binding energy per nucleon for fission products is higher and their total nucleonic mass is lower. The result of this higher binding energy and lower mass results in the production of energy. Essentially, mass defect and nuclear binding energy are interchangeable terms.

The energy released immediately after the fission process is equal to the energy equivalent of the mass lost in the fission process. It can be calculated as given below.

$$n_0^1 +_{92}^{235} U_{92}^{236}U *_{56}^{144} Ba +_{36}^{89} Kr + 3n_0^1 + 210MeV$$

Mass of Uranium
$$m\left({}^{235}_{92}U\right) = 235.043924amu$$

Mass of Neutron
$$m\left(n_0^1\right) = 1.008665amu$$

Mass of Reactants
$$\begin{aligned} m_r &= m\left({}^{235}_{92}U\right) + m\left(n_0^1\right) \\ &= 235.043924 + 1.008665 \\ m_r &= 236.052589amu \end{aligned}$$

Mass of Barium
$$m\left({}^{144}_{56}Ba\right) = 143.922953amu$$

Mass of Krypton
$$m\left({}^{89}_{36}Kr\right) = 88.917631amu$$

Mass of Products
$$\begin{aligned} m_p &= m\left({}^{144}_{56}Ba\right) + m\left({}^{89}_{36}Kr\right) + 3m\left(n_0^1\right) \\ &= 143.922953 + 88.917631 + 3(1.008665) \\ m_p &= 235.866579amu \end{aligned}$$

Mass Defect
$$\begin{aligned} \Delta m &= m_r - m_p \\ &= 236.052589 - 235.866579 \\ \Delta m &= 0.186010amu \end{aligned}$$

This mass defect can be converted to an energy equivalent

$$\Delta E = \Delta mc^2 = 0.186010 \times 931MeV = 173.2MeV$$

3.8.2 The Mass-Energy in Pair Production

Pair production is the creation of an elementary particle and its antiparticle from a photon.

Examples of pair production is creation of an electron and a positron or a proton and an antiproton. Pair production often refers specifically to a photon creating an electron-positron pair near a nucleus. In order to create pair of particle and antiparticle, the incoming energy of the photon must be greater than a threshold energy or at least equal to the total rest mass energy of the two particles, i.e., 1.02. The created particles shall have opposite values of each quantum number. For instance, if one particle has electric charge of $+1$ the other must have electric charge of -1.

The probability of pair production in photon-matter interactions increases with photon energy and also increases approximately as the square of atomic number of the nearby atom.

$$\gamma \longrightarrow e^- + e^+$$

In the process, the energy radiated is converted into matter (electron-positron pair) in accordance with

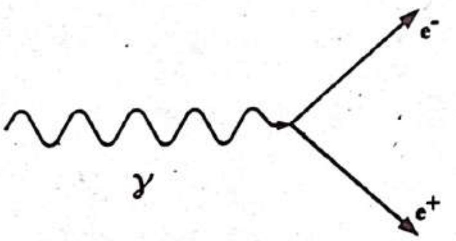


Figure 3.5: *Pair Production*

Einstein's mass energy equivalence $E = mc^2$. For an electron or positron the rest mass energy is

$$\begin{aligned} E_o &= m_o c^2 \\ &= \frac{(9.1 \times 10^{-31})(3 \times 10^8)^2}{1.6 \times 10^{-19}} eV = 0.51 MeV \end{aligned}$$

To create two particles energy required is

$$E_o = 2 \times 0.51 MeV = 1.02 MeV$$

Energy changes in the whole process can be written as

$$hf = 2E_o + K.E.\,(e^-) + K.E.\,(e^+)$$

Where hf is the threshold energy, E_o is the rest mass energy and $K.E.$ are kinetic energies of electron and positron.

3.8.3 The Mass-Energy in Pair Annihilation

The process in which a subatomic particle collides with its respective antiparticle to produce two photons is called Pair annihilation.

Example of pair annihilation is collision of an electron and a positron to produce a pair of photons. The total energy and momentum of the initial pair are conserved in the process.

$$e^+ e^- \longrightarrow \gamma + \gamma$$

When a low energy electron annihilates a low energy positron, they only produce two photons. The creation of only one photon is forbidden by momentum conservation because a single photon would carry nonzero momentum in any frame.

In the annihilation process, electron and positron particles have a rest energy of about $0.511 MeV$. If their kinetic energies are relatively negligible, total energy will be equal to rest energy of the photons produced. Each of the photons then has an energy of about $0.511 MeV$.

In the process, the electron positron pair is converted into energy in accordance with Einstein's

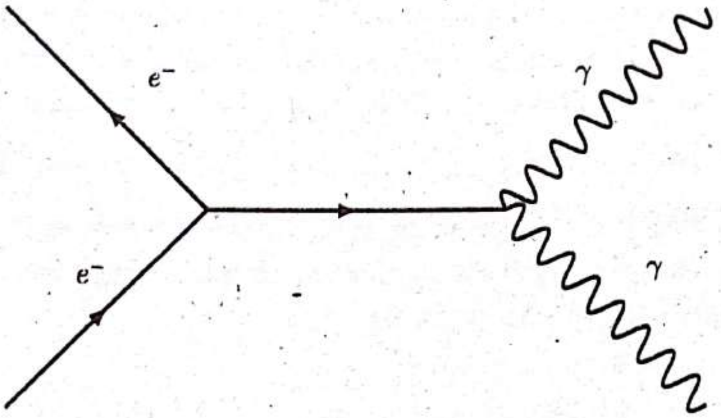


Figure 3.6: *Pair Annihilation*

mass energy equivalence $E = mc^2$. For an electron or positron the rest mass energy is

$$\begin{aligned} E_o &= m_o c^2 \\ &= \frac{(9.1\times10^{-31})(3\times10^8)^2}{1.6\times10^{-19}} eV = 0.51 MeV \end{aligned}$$

To create two particles energy required is

$$E_o = 2 \times 0.51 MeV = 1.02 MeV$$

Energy changes in the whole process can be written as

$$hf = 2E_o + K.E.\,(e^-) + K.E.\,(e^+)$$

The first term represents the rest energy of both particles, the second and third terms are the kinetic energies of electron and positron just before the collision, and the term on the right hand side of the equation represents the creation of two photons, each having a the same frequency f and energy hf . The equation indicates that hf must be at least equal to $E_o = 0.51 MeV$.

3.8.4 Practical Examples of Mass-Energy Equivalence

Whenever energy is added to a system, the system gains mass according to $E = mc^2$. Some of the examples are given below:

- A spring's mass increases whenever it is put into compression or tension. Its added mass arises from the added potential energy stored within it, which is bound in the stretched chemical (electron) bonds linking the atoms within the spring.
- Raising the temperature of an object (increasing its heat energy) increases its mass. For example, consider the world's primary mass standard for the kilogram, made of platinum or iridium. If its temperature is allowed to change by $1°C$, its mass changes by $1.5pg$ ($1pg = 1\times10^{-12}g$).
- A spinning ball weighs more than a ball that is not spinning. Its increase of mass is exactly the

equivalent of the mass of energy of rotation, which is itself the sum of the kinetic energies of all the moving parts of the ball. For example, the Earth itself is more massive due to its daily rotation, than it would be with no rotation. This rotational energy ($2.14 \times 10^{29} J$) represents 2.38 billion metric tons of added mass.

Note that no net mass or energy is really created or lost in any of these examples. Mass and energy simply moves from one place to another. These are some examples of the transfer of energy and mass in accordance with the principle of mass–energy conservation.

3.9 Energy Momentum Relation

Q. Derive energy-momentum relation.

A relativistic equation relating any particle's rest mass, total energy and momentum is called energy momentum relation.

From Einstein's mass energy equivalence relation

$$E = mc^2 \tag{3.14}$$

and relativistic mass is given by

$$m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{3.15}$$

Putting value of m in Eq.(3.14)

$$E = \frac{m_o c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$E^2 = \frac{m_o^2 c^4}{\left(1 - \frac{v^2}{c^2}\right)} \tag{3.16}$$

Momentum is given by

$$p = mv$$

Using Eq.(3.15)

$$p = \frac{m_o v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$p^2 = \frac{m_o^2 v^2}{\left(1 - \frac{v^2}{c^2}\right)}$$

$$p^2c^2 = \frac{m_o^2v^2c^2}{\left(1-\frac{v^2}{c^2}\right)} \tag{3.17}$$

Subtracting Eq.(3.17) from (3.16)

$$\begin{aligned}
E^2 - p^2c^2 &= \frac{m_o^2c^4}{\left(1-\frac{v^2}{c^2}\right)} - \frac{m_o^2v^2c^2}{\left(1-\frac{v^2}{c^2}\right)} \\
&= \frac{m_o^2c^4 - m_o^2v^2c^2}{\left(1-\frac{v^2}{c^2}\right)} \\
&= \frac{m_o^2c^4\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{v^2}{c^2}\right)} \\
E^2 - p^2c^2 &= m_o^2c^4 \\
E^2 &= p^2c^2 + m_o^2c^4
\end{aligned}$$

Where m_o is the rest mass of an object, E is the total energy, c is the speed of light and p is the momentum. This relation is consistent with mass-energy relation $E = mc^2$.

- For a massless particle $m_o = 0$ then above equation reduces to

$$\begin{aligned}
E^2 &= p^2c^2 + (0)^2c^4 \\
E &= pc
\end{aligned}$$

- If $v \ll c$ then energy-momentum relation becomes

$$E = \frac{1}{2}m_ov^2 + m_oc^2$$

- If body is at rest, i.e., $v = 0$ then

$$E = E_o \quad \text{and} \quad m = m_o$$

3.10 The Conservation of Energy and Momentum

Q. Prove the conservation of energy and momentum.

Total relativistic energy is given by

$$\begin{aligned}
E_{\text{total}} &= KE + E_{\text{rest}} \\
\gamma mc^2 &= K.E. + mc^2
\end{aligned}$$

Since mass and energy are not independent quantities, their separate conservation principles are properly a single one, the principle of conservation of mass energy. Mass can be created or destroyed, but when this happens, an equivalent amount of energy simultaneously vanishes or comes into being, and vice versa. Mass and energy are different aspects of the same thing.

Kinetic energy does not have to be conserved in an inelastic collision. Likewise, mass does not have to be conserved since it can be converted into energy. However, the total energy (kinetic, rest mass, and all other potential energy forms) is always conserved in Special Relativity. Momentum and energy are conserved r both elastic and inelastic collisions when the relativistic definitions are used.

3.11 ero Rest Mass Particles

Q. What are zero rest mass particles. Give its examples.

There are three possibilities for classification of relativistic particles:

(i) $E^2 - c^2p^2 > 0 \quad \Rightarrow \quad m_o > 0$, hyperbola in light cone, $v < c$, (particles in daily life)

(ii) $E^2 - c^2p^2 = 0 \quad \Rightarrow \quad m_o = 0$, degenerate limit of hyperbola in light cone, $v = c$, (photons, neutrinos)

(iii) $E^2 - c^2p^2 < 0 \quad \Rightarrow \quad m_o < 0, v > c$, (tachyon)

Those particles whose invariant mass is zero are called massless particles.

There are two known massless particles which are photons and family of neutrinos.

3.11.1 Photons

A photon is a massless and chargeless particle. In 1905, Einstein published a paper in which he proposed many phenomena relating to photon including the photoelectric effect and black body radiations.

According to momentum-energy relation

$$E^2 = p^2c^2 + m^2c^4$$

For photon $m = 0$ (This shows that the rest mass of a particle moving with velocity of light must be zero.)

$$\begin{aligned} E^2 &= p^2c^2 + (0)^2c^4 \\ E^2 &= p^2c^2 \\ E &= pc \end{aligned}$$

According to mass-energy equivalence formula $E = mc^2$. So

$$\begin{aligned} pc &= mc^2 \\ p &= mc \\ mv &= mc \end{aligned}$$

$$v = c$$

In vacuum the photon moves with the speed of light.

Massless particles always travel at the speed of light. Experimental theories also imply that the photon is massless but their momentum is non-zero. This is the reason why light bends due to gravity while having zero mass.

If photon is not massless it would not move with the speed of light and require the modification of Coulomb's law and need an extra degree of freedom. The speed of light is actually constant of nature and is upper limit on speed of any object.

3.11.2 Neutrinos

A neutrino is an elementary particle with half-integer spin and that interacts only via the weak subatomic force and gravity.

The neutrino is so named because it is electrically neutral and because its rest mass is so small (-ino) that it was long thought to be zero. Neutrinos are of three types including electron neutrinos (ν_e), muon neutrinos (ν_μ), and tau neutrinos (ν_τ).

Although neutrinos were long believed to be massless but now it is known that there are three discrete neutrino masses with different tiny values. The mass of the neutrino is much smaller than that of the other known elementary particles. 2016 Cosmological observations imply that the sum of the three masses must be less than one millionth that of the electron ($\leq 0.120eV/c^2$.).

Before neutrinos were found to oscillate, they were assumed to be massless, propagating at the speed of light. According to special relativity, the question of neutrino velocity is closely related to their mass.

If neutrinos are massless, they must travel at the speed of light, and if they have mass they cannot reach the speed of light.

Due to their tiny mass, the predicted speed is extremely close to the speed of light in all experiments, and current detectors are not sensitive to detect the expected difference in the speed of light and speed of neutrinos.

3.11.3 Tachyons

A hypothetical particle which moves with speed greater than speed of light is called tachyon.

Most physicists believe that faster than light particles cannot exist because they are inconsistent with the laws of physics. If such particles did exist, they could be used to build a tachyonic antitelephone and send signals faster than light, which would lead to violations of causality.

Furthermore no experimental evidence for the existence of such particles has been found. These are also called meta particle. The relativistic energy is given by

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

When $v > c$, the denominator in the equation for the energy is "imaginary", as the value under the radical is negative. Because the total energy must be real, the numerator must also be imaginary: i.e. the mass m of the tachyons must be imaginary.

Any particle that could travel faster than speed of light would move backward in time.

3.12 Cherenkov Effect and Cherenkov Radiation

Q. What is Cherenkov effect and Cherenkov radiation. Explain.

An electromagnetic radiation emitted when a charged particle passes through a dielectric medium at a speed greater than the phase velocity of light in that medium is called Cherenkov radiation and such a phenomenon is called Cherenkov Effect.

In 1934, Pavel Chrenkov observed a faint blue light emanating from water that was absorbing radiation. At first he presumed this to be the well-established phenomenon of luminescence, but on further investigation he discovered by passing gamma rays through different liquids that the light was not a function of the medium, as with luminescence, but rather a function of the radiation. In 1937 Ilya Mikhailovich Frank and Icor Evgenievich Tamm ascertained, by developiong and testing a mathematical theory as proff, that the light was emitted by radiaoactive particles moving faster than the speed of light thourgh liquid. The trio shared the 1958 Nobel Prize in physics for their identification and verification of what was called Cherenkov radiation. Understanding of this phenomenon led to the development of the Cherenkov counter, which measures hight-energy particles and gained wide usage in physics experiments measuring Cherenkov radation.

Cherenkov gauged the light output from the radiated liquids with uprotected eyes, so he increased his sensitivity to light by blinding his eyes from light for an hour beform commencing the experiments, which he performed repeatedly. Using elementary laboratory equipment, Cherenkov exposed weak gamma rays to different liquids and observed the mysterious blue light in every trial, thus verifying the radiation and not the liquids as the source of the light. Furthermore Cherenkov differentiaed this radation from luminescence, which dimmed with additives and was polarized. Frank and Tamm joined Cherenkov in 1937 to explain the source of the blue light, which they hypothesized was the visual equivalent of a sonic boom in sound. In other words, light travels fastes in a vacuum, but in a liquid, light travels slower, and this radiation proved fast enough to overtake light. The characteristic blue glow of an underwater nuclear reactor is due to Cherenkov radiation.

The immediate response to their findings was not promising, but with the escalation of World War

II and the cold war in its wake interest in nuclear physics grew exponentially. Cherenkov counters were commonly used in satellites and balloons studying cosmic rays and in research on radiation under the polar ice caps. In 1946, the trio of Cherenkov, Frank and Tamm won the State Prize from the Soviet government before winning the Nobel Prize. In 1970 Cherenkov himself became a member of the U.S.S.R. Academy of sciences.

3.13 The Invariance of Physical Quantities

Q. What is the invariance of physical quantities. Check the invariance of different physical quantities.

A quantity which has same value in all inertial frames is called invariant quantity.

Some invariant quantities are:

- The speed of light in vacuum, c
- The spacetime interval, $(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$
- The proper time, $(\Delta\tau)^2 = \dfrac{(\Delta s)^2}{c^2}$ between two events
- The mass of a particle, m
- The total energy of a particle, E
- The charge of a particle, q

The laws of physics are invariant under a transformation between two coordinate frames moving at constant velocity with respect to each other. To define physical laws in special relativity invariant quantities are very important.

3.14 The Invariance of Speed of Light

For a particle moving along x-axis the inverse Lorentz velocity transformations are

$$\begin{aligned}
u_x &= \frac{u'_x + v}{1 + \frac{v}{c^2}u'_x} \\
u_y &= \frac{u'_y}{1 + \frac{v}{c^2}u'_x} \\
u_z &= \frac{u'_z}{1 + \frac{v}{c^2}u'_x}
\end{aligned}$$

To check the consistency of above equations with speed of light put $u_x = u = c$ and $u'_x = u'$ for a particle motion parallel to the x-axis. So

$$\begin{aligned} u' &= \frac{c-v}{1-\frac{cv}{c^2}} \\ &= \frac{c-v}{1-\frac{v}{c}} \\ &= \frac{c(c-v)}{(c-v)} \\ u' &= c \end{aligned}$$

Thus a particle is moving with velocity c observed by an observer in frame S and its velocity observed by S' is also c. The velocity of the frame has no effect on the velocity of light. This is a verification of the fact that Lorentz transformation is consistent with the postulate of the constancy of velocity of light and hence in accordance with the results of Michelson-Morley experiment.

3.15 The Invariance of Spacetime Interval

According to Einstein's assumption the speed of light is independent of the velocity of observer. i.e.,

$$c = \frac{\Delta r}{\Delta t} = \frac{\Delta r'}{\Delta t'}$$

or

$$c^2 (\Delta t)^2 = (\Delta r)^2, \qquad c^2 (\Delta t')^2 = (\Delta r')^2$$

$$c^2 (\Delta t)^2 - (\Delta r)^2 = 0, \qquad c^2 (\Delta t')^2 - (\Delta r')^2 = 0$$

Comparing above two equations

$$\begin{aligned} c^2 (\Delta t)^2 - (\Delta r)^2 &= c^2 (\Delta t')^2 - (\Delta r')^2 \\ (c\Delta t)^2 - [(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2] &= (c\Delta t')^2 - [(\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2] \end{aligned}$$

Thus $(c\Delta t)^2 - (\Delta r)^2$ is invariant.

3.16 The Invariance of Mass

Portion of the total mass of an object that is independent of the overall motion of the system is called invariant mass.

According to Lorentz transformation mass varies due to relative motion of inertial reference frame. It is a characteristics of the system's total energy and momentum.

In rest frame, invariant mass of a system is equal to total mass while in moving frames, the total mass of the system is greater than invariant mass. If objects are in relative motion then invariant mass will be different from rest mass.

Those systems whose four momentum is zero has zero invariant mass and called massless.

In particle physics invariant mass m_o is given by relation

$$m_o^2c^4 = E^2 - p^2c^4$$

Invariant, covariant, conserved

Invariant or Lorentz-invariant means the same in all reference frames. Covariant is, strictly, a technical term applied to four-vector quantities, but it is often used to mean invariant when it is the mathematical form of an equation that is invariant. Conserved means not changing with time or the same before and after. Rest mass is Lorentz-invariant but not conserved. Energy is conserved but not Lorentz-invariant.

KEY TERMS

- **Mass Variation:** Mass of an object increases due to relative motion.
- **Relativistic Mass:** The mass of an object which changes with speed of the object as its speed approaches to the speed of light is called relativistic mass.
- **Relativistic Momentum:** Relativistic momentum of an object increases due to relative motion.
- **Relativistic Force:** Time rate of change of relativistic momentum is called relativistic force.
- **Mass-Energy Equivalence:** Mass and energy are equivalent forms of each other.

KEY FORMULAE

Mass variation

$$m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Relativistic mass

$$m_r = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma m_o$$

Relativistic momentum

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Relativistic force

$$\mathbf{F} = m\gamma\mathbf{a} + m\gamma^3\left(\frac{\mathbf{v}}{c^2}\cdot\mathbf{a}\right)\mathbf{v}$$

Mass energy equivalence

$$E = mc^2$$

Relativistic kinetic energy

$$K.E. = m_o c^2(\gamma - 1)$$

Total relativistic energy

$$E_{total} = K.E. + E_{rest}$$

Energy-momentum relation

$$E^2 = p^2c^2 + m_o^2c^4$$

EXERCISE

Problems

1. The kinetic energy of a non-relative particle reads $T = \dfrac{p^2}{2m_o}$ where $p = my$ is the momentum of the particle. Find a similar expression for the relativistic kinetic energy.

2. Compute the kinetic energy of a muon (mass $m_\mu = 1.88 \times 10^{-28} kg$) travelling with speed $9c/10$.

3. The proton has mass $m_p = 1.67 \times 10^{-27} kg$. Compute the total (relativistic) energy of a proton moving with speed $v = 3c/5$.

4. At what speed is the total energy of a particle twice the mass energy?

5. In a nuclear fission of uranium-235 caused by the absorption of a neutron, nuclei of krypton and barium are produced, and three neutrons are emitted. The difference in the total mass of the particles present at the start of the process and those present at the end is $3.08 \times 10^{-28} kg$. What is the energy released in this process, in both joules and electronvolts ($1eV = 1.60 \times 10^{-19} J$)?

6. Given that the binding energy of a hydrogen atom is 13.6 eV, what is the difference between the mass of a hydrogen atom and the masses of its constituent electron and proton?

7. In frame of reference S, an electron moving along the x-axis has energy $3m_ec^2$ and momentum magnitude $\sqrt{8}m_ec$. Use the transformations of energy and momentum to find the energy and momentum magnitude observed in frame S' moving with speed $4c/5$ relative to S in the positive x-direction.

8. A particle of mass m_1 moving with speed v, collides with another particle at rest mass m_2. Show that the rest mass M of the combined system is

$$M^2 = m_1^2 + m_2^2 + \frac{2m_1m_2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

9. A stationary atom of proper mass m is struck by a photon of frequency ν and recoils. If the atom absorbs the photon, what are the speed and mass of the recoiling particles?

Chapter 4

Spacetime Structure

Learning Objectives

In this chapter you will learn

- About 3D Euclidean space
- About Minkowski spacetime
- Light cone
- World line
- Spacetime interval
- Lorentz transformations and the spacetime diagram
- Four vectors and examples
- Four potential
- Electromagnetic field tensor
- Lorentz force
- Covariant form of Maxwell's equation

KEY TERMS

Euclidean space, Minkowski spacetime, light cone, world line, spacetime interval, rapidity, covariant and contravariant vector, mixed vectors, Einstein's summation convention, position four vector, four vector derivative, velocity, acceleration, momentum, force, potential, covariant current density, Lorentz force.

A four dimensional model of coordinates which can be used to visualise an event occurrence is called spacetime.

There are two common spacetime structures are commonly used which are described below.

4.1 Three Dimensional Euclidean Space

Q. Describe three dimensional Euclidean sapce.

Three dimensional space in which the axioms and postulates of Euclidean geometry can be applied is called three dimensional Euclidean space.

Until 20th century it was assumed that space and time are independent of each other. But after Lorentz transformations, it was realized that space and time coordinates are not independent coordinates as can be seen in the following equations:

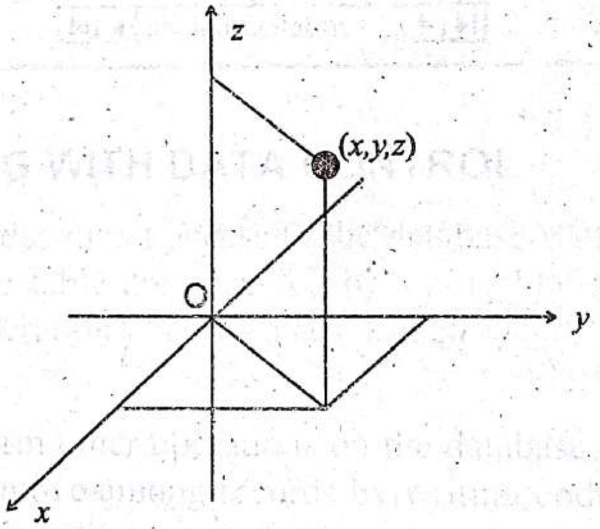


Figure 4.1: *3D Euclidean Space*

$$\begin{aligned} x' &= \gamma(x - vt) \\ y &= y' \\ z &= z' \\ t' &= \gamma\left(1 - \frac{v}{c^2}x\right) \end{aligned}$$

To completely understand the geometry of such equations another structure is required which consists of space and time coordinates. i.e., spacetime coordinates. In one dimension Euclidean space reduces to a line; in two dimensions it reduces to Cartesian plane and in three dimensions it becomes coordinate space (Fig:4.1).

4.2 Minkowski Spacetime

Q. Define and explain Minkowski spacetime

Unification of three dimensional Euclidean space and time coordinate is called Minkowski space.

Minkowski space is a four dimensional space. In 1905, Einstein presented his theory of relativity and in 1908, Minkowski presented a geometric interpretation of special relativity and first time introduced four dimensional continuum known as Minkowski spacetime (Fig:4.2). It is also called flat spacetime. This theory was developed in a way such that spacetime interval is same in different inertial frames of references.

In spacetime each location has is own four coordinates (three spatial and one temporal). The time can be measured by clock placed at each own reference frame.

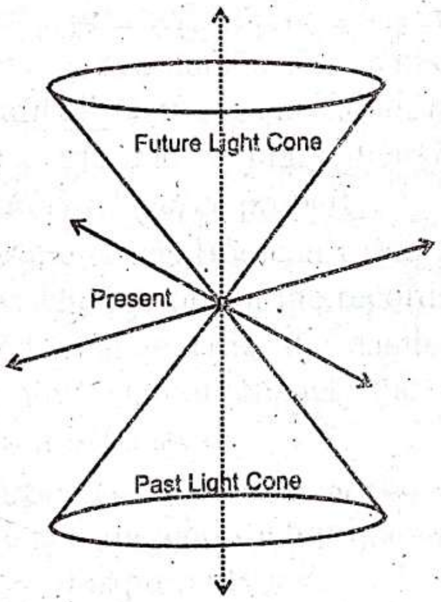


Figure 4.2: *Minkowski Spacetime*

4.2.1 Light Cone

The path adopted by a flash of light travelling in all directions during an event in spacetime is called light cone.

If light is confined to two dimensional plane, the light will diverge in a circular path. This circle alongwith vertical time axis forms a cone called **future light cone**. Light may also converge rather than diverging in a circular path. This circle alongwith vertical time axis form another cone called **past light cone**. The behaviour of past light cone is reverse of the future light cone because past light cone converge for an event and future light cone diverge for an event (Fig:4.2).

Light rays move on the lines $t = x$ and $t = -x$. These lines define the light cone while the origin is some event E in spacetime. The inside of the light cone in the lower half plane is the past of E.

- **The past:** This is the region of space-time between the negative time axis and the lightcone. Events at the origin can be causally connected with events that occur in the past lightcone.
- **The future:** This is the region of space-time between the positive time axis and the lightcone. All events inside the future lightcone could be affected by events at the origin. Any event in the future lightcone can be causally connected with events occurring at (x, y, z) at or after $t = 0$.

An event can be described absolutely by position of the event in spacetime. In Cartesian coordinate system it is convenient to represent an event but the time cannot be represented. An event can be described completely only in Minkowski spacetime. Events occurred in past are represented in past light cone. Event occurring presently are represented at present while events to be occur in future can be represented by future light cone.

Actually light is not confined to two dimensional plane but light travels in three dimensional space. So light will form a diverging or converging sphere rather than circles. Thus light cone will actually be four dimensional Minkowski spacetime structure rather than three dimensional space (Fig:4.2).

More the volume of the lightcone, faster will be the speed of the particle and vice versa.

An event occurred will be observed at same location a time in light cone for all inertial frames but two events at different location occurring at same time will have different past and future light cones.

Motion of a particle is restricted to occur only inside the light cone. The region inside the light cone is called timelike. Regions outside the light cone, which are casually unrelated to the event E, are called spacelike.

4.2.2 World Line

The path of a particle during motion through spacetime as represented in a spacetime diagram is called a worldline.

The concept of worldline was introduced by Hermann Minkowski. It is used to visualize motion of a particle in a Minkowski diagram. This path (worldline) is different from an ordinary path in a sense that ordinary path is in three dimensions but worldline in four dimensions including time dimension. In light cone worldline is a point at present and it fomr a timelike curve in spacetime. Each point on the worldline represents an event with its own space and time coordinates. e.g.,

The worldline of Earth is a circular orbit around the earth locally in solar system but at global level solar system is also changing its position. Thus worldline of earth will be a helix rather than a circle.

So worldline is a curve which tells about the history of a particle travel in spacetime. This curve depends on a parameter called proper time τ. The tangent to the worldline is called four velocity. The worldline goes away from time axis as the speed increases.

4.2.3 Spacetime Interval

The distance between two events in space and time is called spacetime interval.

In Galilean transformation, time and space are absolute quantities but in Lorentz transformation, time and space are no longer absolute quantities. However spacetime interval in invariant and it is the basic requirement of special theory of relativity. i.e.,

$$\begin{aligned}(\Delta s)^2 &= (\Delta s')^2 \\ (ct)^2 - x^2 - y^2 - z^2 &= (ct')^2 - x'^2 - y'^2 - z'^2\end{aligned}$$

To prove invariance of spacetime interval, apply Lorentz transformations on right side of above equation

$$\begin{aligned}(\Delta s')^2 &= (ct')^2 - x'^2 - y'^2 - z'^2 \\ &= c^2\left[\gamma\left(t - \frac{v}{c^2}x\right)\right]^2 - [\gamma(x - vt)]^2 - y^2 - z^2 \\ &= c^2\gamma^2\left(t - \frac{v}{c^2}x\right)^2 - \gamma^2(x - vt)^2 - y^2 - z^2 \\ &= c^2\gamma^2\left(t^2 + \frac{v^2}{c^4}x^2 - 2\frac{v}{c^2}xt\right) - \gamma^2\left(x^2 + v^2t^2 - 2vxt\right) - y^2 - z^2 \\ &= \gamma^2\left[c^2t^2 + \frac{v^2}{c^2}x^2 - 2vxt - x^2 - v^2t^2 + 2vxt\right] - y^2 - z^2 \\ &= \gamma^2\left[(c^2t^2 - x^2) - \left(v^2t^2 - \frac{v^2}{c^2}x^2\right)\right] - y^2 - z^2 \\ &= \gamma^2\left[(c^2t^2 - x^2) - \frac{v^2}{c^2}(c^2t^2 - x^2)\right] - y^2 - z^2 \\ &= \gamma^2\left(1 - \frac{v^2}{c^2}\right)(c^2t^2 - x^2) - y^2 - z^2 \\ &= \frac{1}{\left(1 - \frac{v^2}{c^2}\right)}\left(1 - \frac{v^2}{c^2}\right)(c^2t^2 - x^2) - y^2 - z^2 \\ &= c^2t^2 - x^2 - y^2 - z^2 \\ (\Delta s')^2 &= (\Delta s)^2\end{aligned}$$

Thus spacetime interval has the same values in all inertial frames. As speed of light is Lorentz invariant, so equivalently it can be stated that c and s are absolute. It is invariant under rotation of the coordinate system and can be related with proper time as

$$(\Delta\tau)^2 = \frac{(\Delta s)^2}{c^2}$$

Spacetime interval may have three different values:

- If $(\Delta s)^2 > 0$ it is called timelike; time between the events is long enough that particles moving with speed $v < c$ could move from one event to the other. Events are casually related and there exist a frame in which two events happen at same place but at different times.

- If $(\Delta s)^2 < 0$ it is called spacelike; time between the events is too short for any physical influence to move between them. Such events are not casually related and a frame exist in which the two events happen at the same time but at different places.
- If $(\Delta s)^2 = 0$ it is called lightlike; light signal could travel directly from on event to the other. Such events are casually related an all observers will agree that they could be linked by a light signal.

4.3 Lorentz Transformation and the Spacetime Diagram

Q. What are Lorentz transformation and relate it with spacetime diagram.

Consider an event is being observed in two frames of reference S and S' in relative motion with each other. In spacetime diagram $x-$ and $ct-$axes are drawn for S' frame while $x'-$ and $ct'-$ axes are drawn for S' frame. $x-$axis and $ct-$axis shows the spatial coordinates in the stationary frame and moving frame respectively. Similarly $ct-$axis and $ct'-$axis shows the temporal (time) coordinates in the stationery and moving frame respectively (Fig.4.3).

For $x'-$axis take $x' = 0$ and $t' = 0$ in Lorentz transformations.

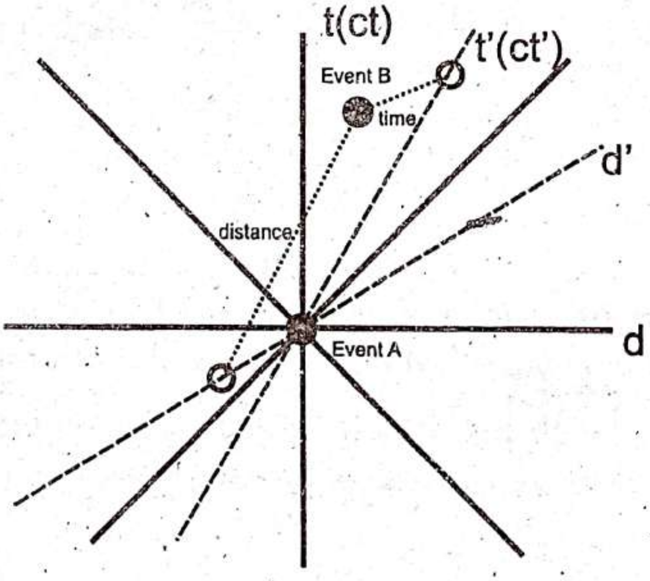


Figure 4.3: *Spacetime coordinates in the Stationery and Moving frame*

$$x' = \gamma(x - vt) \qquad \Rightarrow \quad x = vt$$
$$t' = \gamma\left(t - \frac{v}{c}x\right) \qquad \Rightarrow \quad t = \frac{v}{c}x$$

From above equations it can be inferred that $x'-$axis is represented by the line $x = vt$; a straight line passing through the origin with gradient $\frac{1}{v}$ in the spacetime diagram of S. Similarly $t'-$axis is represented by the line $t = \frac{v}{c}x$; a straight line passing through the origin with gradient $\frac{v}{c}$. The slope of the worldline can also be determined using angle θ by

$$\tan\theta = \frac{v}{c}$$

If velocity v increases $ct'-$ and $x'-$axes moves away from $ct-$ and $x-$axes respectively and vice versa. The line $x = ct$ is the path of light ray which gives the information of all the events.

4.3.1 Lorentz transformation as a rotation in 4D spacetime

Lorentz transformation equations are given by

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(1 - \frac{v}{c^2}x\right) \end{aligned}$$

The velocity v of a moving frame is directly proportional to the tangent of the angle θ as can be seen form Fig.(4.4)

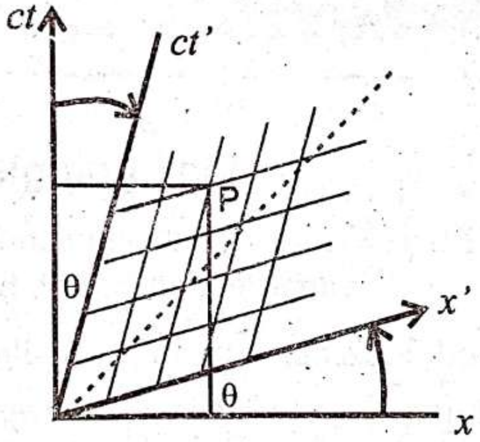


Figure 4.4: *Lorentz rotation in the spacetime diagram*

$$\begin{aligned} v &\propto \tanh\theta \\ v &= c\tanh\theta \end{aligned}$$

Where θ is the parameter which relates v and c called rapidity. It is basically hyperbolic angle of rotation in Minkowski space.

From above equation

$$\frac{v}{c} = \tanh\theta$$

So Lorentz transformation equations becomes

$$\begin{aligned} x' &= \frac{x - ct\,(\tanh\theta)}{\sqrt{1 - \tanh^2\theta}} \\ &= \frac{x - ct(\tanh\theta)}{\sqrt{sech^2\theta}} \\ &= \frac{x - ct\,(\tanh\theta)}{sech\theta} \end{aligned}$$

$$
\begin{aligned}
&= x\cosh\theta - ct\frac{\sinh\theta}{\cosh\theta}\cosh\theta \\
x' &= x\cosh\theta - ct\sinh\theta
\end{aligned}
$$

and

$$
\begin{aligned}
t' &= \frac{t - x\dfrac{c\tanh\theta}{c^2}}{\sqrt{1-\tanh^2\theta}} \\
&= \frac{t-\left(\dfrac{x}{c}\right)\tanh\theta}{\sqrt{sech^2\theta}} \\
&= \frac{t-\left(\dfrac{x}{c}\right)\tanh\theta}{sech\theta} \\
&= t\cosh\theta - \left(\frac{x}{c}\right)\frac{\sinh\theta}{\cosh\theta}\cosh\theta \\
t' &= t\cosh\theta - \left(\frac{x}{c}\right)\sinh\theta \\
ct' &= ct\cosh\theta - x\sinh\theta
\end{aligned}
$$

Thus using rotation through angle θ complete set of Lorentz transformations can be written as

$$
\begin{aligned}
ct' &= (\cosh\theta)ct + (\sinh\theta)x + (0)y + (0)z \\
x' &= (\sinh\theta)ct + (\cosh\theta)x + (0)y + (0)z \\
y' &= (0)ct + (0)x + (1)y + (0)z \\
z' &= (0)ct + (0)x + (0)y + (1)z
\end{aligned}
$$

In matrix form it will be

$$
\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}
$$

$$
X' = \Lambda X
$$

Where Λ is rotation matrix in special relativity.

Therefore the Lorentz transformation can be considered as a rotation through an angle θ in Minkowski space.

4.4 Four Vectors

Q. What are four vectors. Give its typical examples.

Vectors represented in four dimensional Minkowski spacetime are called four vectors.

A four vector can be represented by four coordinates . It is also called Lorentzian vector. A four vector is denoted by $a_\mu(\mu = 0, 1, 2, 3)$ or a^μ.

4.4.1 Contravariant Vectors

When we increased the lengths of the basis vectors the components of our vector decreased if we decrease the lengths of the basis vectors the components of our vector increase because these two quantities change contrary to one another we refer to this by saying that these are the contravariant components of the vector.

4.4.2 Covariant Vectors

Suppose we describe the vector in terms of its dot product with each of the basis vectors. If the basis vectors are of length ℓ then the result of these dot products will be these three lengths if we double the length of a basis vector then it's associated dot product will also double when we increase the lengths of the basis vectors the dot products also increase when we decrease the lengths of the basis vectors the dot products also decrease because both quantities vary in the same way we refer to this by saying that these are the covariant components.

4.4.3 Mixed Vectors

A four vector which is neither contravariant nor covariant is called mixed vector.

For example, δ_α^β is a mixed vector.

In matrices a_μ is a covariant four vector represented by a row matrix and a^μ is a covariant four vector represented by a column matrix. Common notation of a covariant four vector is

$$a_\mu = (a^0, a^1, a^2, a^3)$$

and for a contravariant four vector is

$$a^\mu = (a^0, a^1, a^2, a^3)$$

In matrix form, contravariant and covariant four vectors respectively can be written as

$$a^\mu = \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}, \qquad a_\mu = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \end{pmatrix}$$

The inner (scalar) product of two four vectors is given by

$$a \cdot b = g_{\mu\nu} a^{\mu} b^{\nu}$$

Where $g_{\mu\nu}$ is Minkowski spacetime metric given by

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \text{diag}\,(-1, 1, 1, 1)$$

In matrix form the inner product will be

$$\begin{aligned} a^{\mu} b_{\mu} &= \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix} \begin{pmatrix} b_0 & b_1 & b_2 & b_3 \end{pmatrix} \\ a^{\mu} b_{\mu} &= a^0 b_0 + a^1 b_1 + a^2 b_2 + a^3 b_3 \end{aligned}$$

In special relativity four vectors transform under Lorentz transformation as

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$$

Where Λ^{μ}_{ν} is the Lorentz matrix given by

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} \Lambda^0_0 & \Lambda^0_1 & \Lambda^0_2 & \Lambda^0_3 \\ \Lambda^1_0 & \Lambda^1_1 & \Lambda^1_2 & \Lambda^1_3 \\ \Lambda^2_0 & \Lambda^2_1 & \Lambda^2_2 & \Lambda^2_3 \\ \Lambda^3_0 & \Lambda^3_1 & \Lambda^3_2 & \Lambda^3_3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Lorentz transformation leaves square length invariant. i.e.,

$$x'^{\mu} x'_{\mu} = x^{\mu} x_{\mu}$$

4.4.4 Raising and Lowering Indices

There is a contravariant four vector (a^{μ}) to each covariant four vector (a_{μ}) which can be interchanged using Minkowski matrix $(\eta_{\mu\nu})$. i.e.,

$$a_{\mu} = \eta_{\mu\nu} a^{\nu}$$

where $(\eta_{\mu\nu})$ is Minkowski matrix (metric) given by

$$\eta_{\mu\nu} = \begin{pmatrix} \eta_{00} & \eta_{01} & \eta_{02} & \eta_{03} \\ \eta_{10} & \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{20} & \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{30} & \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta^{\mu\nu}$$

Above expression shows lowering of index. Thus Minkowski matrix can be used to lower the indices of four vectors.

Same can be done to interchange covariant four vector to contravariant four vectors. i.e.,

$$a^{\mu} = \eta^{\mu\nu} a_{v}$$

where $\eta^{\mu\nu}$ is also Minkowski matrix (metric).

Since $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ have identical components, these two can be related as

$$\begin{aligned} \eta_{\alpha\nu}\eta^{\nu\beta} = \eta^{\alpha\nu}\eta_{\nu\beta} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ \eta_{\alpha\nu}\eta^{\nu\beta} &= \delta^{\alpha}_{\beta} = \delta^{\beta}_{\alpha} \end{aligned}$$

where δ^{β}_{α} is Kronecker delta function given by

$$\delta^{\beta}_{\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4.4.5 Einstein Summation Convention

Indices appeared twice in an expression are written as

$$\sum_{\mu} a_{\mu} b^{\mu} = a_{\mu} b^{\mu}$$

On the right hand side there is no summation written. It will be understood that summation is present. This is called summation convention. This convention can only be applied to expressions having same indices.

For two indices, same expression can be written as

$$\sum_{\mu\nu} a_{\mu\nu} b^{\mu} b^{\nu} = a_{\mu\nu} b^{\mu} b^{\nu}$$

The repeated indices, over which a summation is understood are called dummy indices. These indices can be replaced by any other index. The same index must not be used more than twice in any single expression. For example, $\sum_{i=1}^{N} \left(a_i x^i\right)^2$ cannot be written as $a_i x^i a_i x^i$ rather it must be written as $a_i x^i a_j x^j$. For two or more dummy indices different letters are used.

Those indices which are not summed over are called free indices. For example in the following expression μ is free index

$$a^{\mu} = g^{\mu\nu} a_{\nu}$$

4.5 Position Four Vector

Q. What is position four vector. Describe in details.

A four vector which gives the displacement of an event from origin in Minkowski space is called position four vector.

It is given by

$$x^{\mu} = \left(x^{o}, x^{i}\right) = \left(x^{o}, \mathbf{x}\right) = \left(ct, x, y, z\right)$$

Position four vector in covariant form is

$$x_{\mu} = \left(x_o, x_i\right) = \left(x_o, -\mathbf{x}\right) = \left(ct, -x, -y, -z\right)$$

By comparison with position four vector in contravariant form it can be seen that

$$x_o = x^o \qquad \text{and} \qquad x_i = x^i = \mathbf{x}$$

In matrix form, position four vector is

$$x^{\mu} = \begin{pmatrix} x^o \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \qquad x_{\mu} = \begin{pmatrix} x_o & x_1 & x_2 & x_3 \end{pmatrix} = \begin{pmatrix} ct & -x & -y & -z \end{pmatrix}$$

For displacement four vector between two events in Minkowski space following set of coordinates is used

$$\Delta x^{\mu} = \left(c\Delta t, \Delta x, \Delta y, \Delta z\right)$$

The inner product of position four vector will be

$$\begin{aligned}
x^\mu x_\mu &= x^o x_o + x^1 x_1 + x^2 x_2 + x^3 x_3 \\
&= (ct)(ct) + (x)(-x) + (y)(-y) + (z)(-z) \\
&= (ct)^2 - x^2 - y^2 - z^2 \\
&= (ct)^2 - (x^2 + y^2 + z^2) \\
x^\mu x_\mu &= (ct)^2 - (\mathbf{x})^2
\end{aligned}$$

Spacetime interval $(\Delta s)^2$ can also be calculated using differential position four vectors as

$$(\Delta s)^2 = g_{\mu\nu} \Delta x^\mu \Delta x^\nu$$

Where $g_{\mu\nu}$ is metric in spacetime.

The square of line element between two neighbouring points x^μ and $x^\mu + dx^\mu$ in Minkowski space is given by quadratic differential

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = c^2 dt^2 - d\mathbf{x}^2$$

4.6 Some Typical Four Vectors

Q. What are four derivative, velocity, acceleration, momentum and force. Also find the relation between energy and momentum.

4.6.1 Four Vector Derivative

Four gradient which gives differential change in spacetime coordinates is called four vector derivative.

In special relativity, four vector derivative is also a four vector. The position four vector derivative (analogue to del operator) is given by

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Contraction of contravariant and covariant derivative operator is called D'Almbertian which is given by

$$\begin{aligned}
\partial^\mu \partial_\mu &= \partial^0 \partial_0 + \partial^1 \partial_1 + \partial^2 \partial_3 + \partial^3 \partial_3 \\
&= \left(\frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{1}{c}\frac{\partial}{\partial t}\right) + \left(\frac{\partial}{\partial x}\right)\left(-\frac{\partial}{\partial x}\right) + \left(\frac{\partial}{\partial y}\right)\left(-\frac{\partial}{\partial y}\right) + \left(\frac{\partial}{\partial z}\right)\left(-\frac{\partial}{\partial z}\right) \\
&= \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \\
&= \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)
\end{aligned}$$

$$\partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \equiv \Box$$

Where $\Box$ notation is used for D'Almbertian operator and in special relativity it is analogue of the Laplacian operator (∇^2).

The Lorentz transformations can be used to transform four vector derivative as

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \Lambda^\mu_{\ '\mu}\frac{\partial}{\partial x^\mu}$$

4.6.2 Velocity Four Vector

Time derivative of position four vector is called velocity four vector.

It is given by

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma\frac{dx^\mu}{dt} = \gamma\,(c, \mathbf{v})$$

Where v is the velocity three vector.

Scalar product of four velocity vector is

$$\begin{aligned}
u^\mu u_\mu &= u^0 u_0 + u^1 u_1 + u^2 u_2 + u^3 u_3 \\
&= (\gamma c)(\gamma c) + (\gamma v_x)(-\gamma v_x) + (\gamma v_y)(-\gamma v_y) + (\gamma v_z)(-\gamma v_z) \\
&= \gamma^2 c^2 - \gamma^2 \left(v_x^2 + v_y^2 + v_z^2\right) \\
&= \gamma^2 \left(c^2 - v^2\right) \\
&= \frac{1}{\left(1 - \frac{v^2}{c^2}\right)}\left(c^2 - v^2\right) \\
&= \frac{c^2}{\left(c^2 - v^2\right)}\left(c^2 - v^2\right) \\
u^\mu u_\mu &= c^2
\end{aligned}$$

Like inner product of four position vector, inner product of velocity four vector is also invariant.

4.6.3 Acceleration Four Vector

Proper time derivative of velocity four vector is called acceleration four vector.

Acceleration four vector is given by

$$a^\mu = \frac{du^\mu}{d\tau} = \gamma\frac{du^\mu}{dt} = \gamma^2\left(\gamma^2\frac{\mathbf{u}\cdot\mathbf{a}}{c}, \mathbf{a} + \gamma^2\frac{\mathbf{u}\cdot\mathbf{a}}{c^2}\mathbf{u}\right)$$

Where $\mathbf{a} = \frac{d\mathbf{u}}{dt}$ is ordinary acceleration vector in three dimensional space.

4.6.4 Momentum Four Vector

> Product of scalar mass and velocity four vector is called momentum four vector.

It is given by

$$p^\mu = mu^\mu = m\gamma(c, \mathbf{v})$$

Zeroth component of the momentum is

$$p^0 = \gamma mc$$

In non-relativistic limit $v \ll c, \gamma \to 1$. So

$$p^0 = (1)mc = mc$$

Thus another form of momentum four vector is

$$p^\mu = (p^0, p^i) = (mc, \mathbf{p}) = \left(\frac{E}{c}, \mathbf{p}\right)$$

In natural units $c = 1$, so that $p^o = E$. Thus p^o can be considered as the energy of the particle.

4.6.5 Energy–Momentum Relation

Scalar product of momentum four vector is

$$p^\mu p_\mu = p^0 p_0 + p^1 p_1 + p^2 p_2 + p^3 p_3$$

Using $p^\mu = m\gamma(c, \mathbf{v})$on left hand side and $p^\mu = \left(\frac{E}{c}, \mathbf{p}\right)$ on right hand side

$$\begin{aligned}
(m\gamma c)(m\gamma c) - (m\gamma \mathbf{v})(m\gamma \mathbf{v}) &= \left(\frac{E}{c}\right)\left(\frac{E}{c}\right) - \mathbf{p}\cdot\mathbf{p} \\
\gamma^2 m^2 c^2 - \gamma^2 m^2 v^2 &= \frac{E^2}{c^2} - p^2 \\
\gamma^2 m^2 \left(c^2 - v^2\right) &= \frac{E^2}{c^2} - p^2 \\
\frac{1}{\left(1 - \frac{v^2}{c^2}\right)} m^2 \left(c^2 - v^2\right) &= \frac{E^2}{c^2} - p^2 \\
\frac{c^2}{\left(c^2 - v^2\right)} m^2 \left(c^2 - v^2\right) &= \frac{E^2}{c^2} - p^2 \\
m^2 c^2 &= \frac{E^2}{c^2} - p^2 \\
m^2 c^4 &= E^2 - p^2 c^2 \\
E^2 &= p^2 c^2 + m^2 c^4
\end{aligned}$$

This is called energy-momentum relation.

For photon $m = 0$, so above relation becomes

$$E^2 = p^2c^2$$

or

$$E = pc$$

4.6.6 Force Four Vector

Proper time derivative of momentum four vector is called force four vector.

Classical force is given by

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

Force four vector is given by

$$K^\mu = \frac{dp^\mu}{d\tau} = \gamma\frac{dp^\mu}{dt}$$

As $p^\mu = \left(\frac{E}{c}, \mathbf{p}\right)$, so

$$\begin{aligned} K^\mu &= \gamma\frac{d}{dt}\left(\frac{E}{c}, \mathbf{p}\right) \\ &= \gamma\left(\frac{1}{c}\frac{dE}{dt}, \frac{d\mathbf{p}}{dt}\right) \\ K^\mu &= \gamma\left(\frac{1}{c}\frac{dE}{dt}, \mathbf{F}\right) \end{aligned}$$

In non-relativistic case, $\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{u}$. So

$$K^\mu = \gamma\left(\frac{\mathbf{F}\cdot\mathbf{u}}{c}, \mathbf{F}\right)$$

These are components of covariant four force vector.

4.7 Relativistic Electrodynamics

Q. What is relativistic electrodynamics? Also discuss four potential and covariant current density.

Formulation of physical phenomenon in electrodynamics explained using relativity is called relativistic electrodynamics.

From the results of Michelson Morley experiment and development of special relativity it was found that Newtonian mechanics is not consistent with special relativity. But classical electrodynamics including

Maxwell's equations and the Lorentz force were found consistent with special relativity in any inertial frame.

When an observer interpret a process as an electric then another observer may regard the process as magnetic. Here, what may be the fault with electrodynamics is not actually failure of electrodynamics but a mere formulation of relativistic electrodynamics.

In both classical and relativistic electrodynamics, theory is unchanged rather notations disclose the relativistic character of expressions.

4.7.1 Four Potential

Four dimensional analogue of electromagnetic potential represented in spacetime is called four potential.

It is given by

$$A^\mu = (A^0, A^i) = \left(\frac{V}{c}, \mathbf{A}\right)$$

Where V is scalar potential and $\mathbf{A}$ is vector potential.

4.7.2 Covariant Current Density

Four dimensional analogue of the electric current density represented in spacetime is called four current density.

It is given by

$$J^\mu = (J^0, J^i) = (c\rho, \mathbf{J})$$

Continuity equation in terms of four current density is

$$\partial_\mu J^\mu = 0$$

4.8 Electromagnetic Field Tensor

Q. Write a short note on electromagnetic field tensor. Also derive a relation for Lorentz force. Also write covariant form of Maxwell's equations.

The combination of the electric and magnetic field into a covariant tensor is called electromagnetic field tensor.

Electromagnetic field tensor is also called field strength tensor. This tensor allows to rewrite the laws of electromagnetism in a very concise way. It is, mathematically, exterior derivative of four potential (A^μ). Therefore electromagnetic field tensor $(F_{\mu\nu})$ in Minkowski space can be written as

$$F_{\mu\nu} = \partial_\mu A_\upsilon - \partial_\nu A_\mu$$

Where $\partial)\mu$ is four vector derivative.

Electromagnetic tensor is an antisymmetric tensor. i.e.,

$$F_{\mu\nu} = -F_{\nu\mu}$$

In matrix form, $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \begin{pmatrix} F_{00} & F_{01} & F_{02} & F_{03} \\ F_{10} & F_{11} & F_{12} & F_{13} \\ F_{20} & F_{21} & F_{22} & F_{23} \\ F_{30} & F_{31} & F_{32} & F_{33} \end{pmatrix} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

Using the spacetime metric tensor $g^{\mu\nu}$, contravariant form of electromagnetic field tensor is

$$F^{\mu\nu} = g^{\mu\alpha} F_{\alpha\beta} g^{\beta\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

Another useful tensor is Faraday's dual field tensor $\left(\tilde{F}^{\mu\nu}\right)$ given by

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_x/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix}$$

Components of electric field (E_i) and magnetic field (B_i) can also be easily determined from the electromagnetic field tensor $(F_{\mu\nu})$ using following simple relations

$$E_i = cF_{0i}, \qquad B_i = \frac{1}{2}\varepsilon^{ijk} F_{jk}$$

Where ε^{ijk} is the Levi-Civita symbol given by

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if} ijk \text{is an even permutation of 123,} \\ -1 & \text{if} ijk \text{is an odd permutation of 123,} \\ 0 & \text{if} i = j = k \end{cases}$$

4.8.1 Lorentz Force

In classical electrodynamics, sum of electric and magnetic force is called Lorentz force.

For a charged particle q moving with velocity $\mathbf{v}$, Lorentz force is given by

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Equivalently in components form, it can be written as

$$\begin{aligned} F_x &= q\left[E_x + (\mathbf{v} \times \mathbf{B})_x\right] = q\left(E_x + v_y B_z - v_z B_y\right) \\ F_y &= q\left[E_y + (\mathbf{v} \times \mathbf{B})_y\right] = q\left(E_y - v_x B_z + v_z B_x\right) \\ F_z &= q\left[E_z + (\mathbf{v} \times \mathbf{B})_z\right] = q\left(E_z + v_x B_y - v_y B_x\right) \end{aligned}$$

Or in matrix form,

$$\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = q \begin{pmatrix} E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} c \\ v_x \\ v_y \\ v_z \end{pmatrix}$$

In above expression, on the right hand side, first matrix is something like electromagnetic field tensor $(F_{\mu\nu})$. A similar relation for Lorentz force in spacetime can also be written as

$$K_\mu = q \sum_{\nu=0}^{3} F_{\mu\nu} u^\nu$$

Where u^ν is velocity four vector. As $K_\mu = \gamma\left(\frac{\mathbf{F} \cdot \mathbf{u}}{c}, \mathbf{F}\right)$ and $u^\nu = \gamma\left(c, \mathbf{v}\right)$. So in matrix form Lorentz force is

$$\begin{pmatrix} \gamma\frac{\mathbf{F} \cdot \mathbf{u}}{c} \\ \gamma F_x \\ \gamma F_y \\ \gamma F_z \end{pmatrix} = q \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} c \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix}$$

Using Lorentz transformation in matrix form, electromagnetic field tensor will be

$$F'_{\mu\nu} = \sum_{\alpha,\beta=0}^{3} \Lambda_\mu^\alpha \Lambda_\nu^\beta F_{\alpha\beta}$$

Or equivalently for moving frame S parallel to x-axis, components of electric and magnetic field will be

$$E'_{\parallel} = E_{\parallel} \qquad E'_{\perp} = \gamma\left[E_{\perp} + (\mathbf{v} \times \mathbf{B})_{\perp}\right]$$

$$B'_{\parallel} = B_{\parallel} \qquad B'_{\perp} = \gamma\left[B_{\perp} + \frac{(\mathbf{v}\times\mathbf{E})_{\perp}}{c^2}\right]$$

4.8.2 Covariant Form of Maxwell's Equations

In electrodynamics, Maxwell's equations are written as (in Gaussian units)

$$\left.\begin{aligned}\nabla\cdot\mathbf{E} &= 4\pi\rho\\ \nabla\times\mathbf{B} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} &= \frac{4\pi}{c}\mathbf{J}\end{aligned}\right\}\text{(Homogeneous)}$$

$$\left.\begin{aligned}\nabla\cdot\mathbf{B} &= 0\\ \nabla\times\mathbf{E} + \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} &= 0\end{aligned}\right\}\text{(Non-Homogeneous)}$$

Consider set of homogeneous Maxwell's equations and writing them in the components form as

$$\begin{aligned}
\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} &= 4\pi\rho\\
\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \frac{1}{c}\frac{\partial E_x}{\partial t} &= \frac{4\pi}{c}J_x\\
-\frac{\partial B_z}{\partial x} + \frac{\partial B_x}{\partial z} - \frac{1}{c}\frac{\partial E_y}{\partial t} &= \frac{4\pi}{c}J_y\\
\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} - \frac{1}{c}\frac{\partial E_z}{\partial t} &= \frac{4\pi}{c}J_z
\end{aligned}$$

To write in four vector notation, rewrite above equations in the following form

$$\begin{aligned}
\frac{1}{c}\frac{\partial}{\partial x^0}(0) + \frac{\partial}{\partial x^1}(E_x) + \frac{\partial}{\partial x^2}(E_y) + \frac{\partial}{\partial x^3}(E_z) &= -\frac{4\pi}{c}J^0 &\Rightarrow \partial_\mu F^{0\nu} &= -\frac{4\pi}{c}J^0\\
\frac{1}{c}\frac{\partial}{\partial x^0}(-E_x) + \frac{\partial}{\partial x^1}(0) + \frac{\partial}{\partial x^2}(B_z) + \frac{\partial}{\partial x^3}(-B_y) &= -\frac{4\pi}{c}J^1 &\Rightarrow \partial_\mu F^{1\nu} &= -\frac{4\pi}{c}J^1\\
\frac{1}{c}\frac{\partial}{\partial x^0}(-E_y) + \frac{\partial}{\partial x^1}(-B_z) + \frac{\partial}{\partial x^2}(0) + \frac{\partial}{\partial x^3}(B_x) &= -\frac{4\pi}{c}J^2 &\Rightarrow \partial_\mu F^{2\nu} &= -\frac{4\pi}{c}J^2\\
\frac{1}{c}\frac{\partial}{\partial x^0}(-E_z) + \frac{\partial}{\partial x^1}(B_y) + \frac{\partial}{\partial x^2}(-B_x) + \frac{\partial}{\partial x^3}(0) &= -\frac{4\pi}{c}J^3 &\Rightarrow \partial_\mu F^{3\nu} &= -\frac{4\pi}{c}J^3
\end{aligned}$$

Generalizing above equations

$$\partial_\mu F^{\mu\nu} = -\frac{4\pi}{c}J^\nu$$

Now taking set of inhomogeneous Maxwell's equations and writing them in components form as

$$\begin{aligned}
\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} &= 0\\
\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \frac{1}{c}\frac{\partial B_x}{\partial t} &= 0\\
-\frac{\partial E_z}{\partial x} + \frac{\partial E_x}{\partial z} + \frac{1}{c}\frac{\partial B_y}{\partial t} &= 0
\end{aligned}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \frac{1}{c}\frac{\partial E_z}{\partial t} = 0$$

To write in four vector notation rewrite above equations in the following form

$$\begin{aligned}
\frac{1}{c}\frac{\partial}{\partial x^0}(0) + \frac{\partial}{\partial x^1}(B_x) + \frac{\partial}{\partial x^2}(B_y) + \frac{\partial}{\partial x^3}(B_z) &= 0 \qquad \Rightarrow \partial_\mu \tilde{F}^{0\nu} = 0 \\
\frac{1}{c}\frac{\partial}{\partial x^0}(B_x) + \frac{\partial}{\partial x^1}(0) + \frac{\partial}{\partial x^2}(E_z) + \frac{\partial}{\partial x^3}(-E_y) &= 0 \qquad \Rightarrow \partial_\mu \tilde{F}^{1\nu} = 0 \\
\frac{1}{c}\frac{\partial}{\partial x^0}(B_y) + \frac{\partial}{\partial x^1}(-E_z) + \frac{\partial}{\partial x^2}(0) + \frac{\partial}{\partial x^3}(E_x) &= 0 \qquad \Rightarrow \partial_\mu \tilde{F}^{2\nu} = 0 \\
\frac{1}{c}\frac{\partial}{\partial x^0}(-E_z) + \frac{\partial}{\partial x^1}(B_y) + \frac{\partial}{\partial x^2}(-B_x) + \frac{\partial}{\partial x^3}(0) &= 0 \qquad \Rightarrow \partial_\mu \tilde{F}^{3\nu} = 0
\end{aligned}$$

Generalizing above equations

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

As $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$, so

$$\partial^\mu F^{\nu\alpha} + \partial^\nu F^{\alpha\mu} + \partial^\alpha F^{\mu\nu} = 0$$

Thus Maxwell's equations in covariant form are

$$\begin{aligned}
\partial_\mu F^{\mu\nu} &= -\frac{4\pi}{c}J^\nu \\
\partial_\mu \tilde{F}^{\mu\nu} &= 0
\end{aligned}$$

KEY TERMS

- **Euclidean Space:** Three dimensional space in which the axioms and postulates of Euclidean geometry can be applied is called three dimensional Euclidean space.
- **Minkowski Space:** Unification of three dimensional Euclidean space and time coordinate is called Minkowski space.
- **Light Cone:** The path adopted by a flash of light travelling in all directions during an event in spacetime is called light cone.
- **World Line:** The path of a particle during motion through spacetime as represented in a spacetime diagram is called a worldline.
- **Spacetime Interval:** The distance between two events in space and time is called spacetime interval.
- **Rapidity:** The hyperbolic rotation in Lorentz transformations is called rapidity.
- **Mixed Vector:** A four vector which is neither contravariant nor covariant is called mixed vector.
- **Position Four Vector:** A four vector which gives the displacement of an event from origin in Minkowski space is called position four vector.
- **Four Vector Derivative:** Four gradient which gives differential change in spacetime coordinates is called four vector derivative.
- **Velocity Four Vector:** Time derivative of position four vector is called velocity four vector.
- **Acceleration Four Vector:** Proper time derivative of velocity four vector is called acceleration four vector.
- **Momentum Four Vector:** Product of scalar mass and velocity four vector is called momentum four vector.
- **Force Four Vector:** Proper time derivative of momentum four vector is called force four vector.
- **Four Potential:** Four dimensional analogue of electromagnetic potential represented in spacetime is called four potential.
- **Covariant Current Density:** Four dimensional analogue of the electric current density represented in spacetime is called four current density.
- **Lorentz Force:** In classical electrodynamics, sum of electric and magnetic force is called Lorentz force.
- **Relativistic Electrodynamics:** Formulation of physical phenomenon in electrodynamics explained using relativity is called relativistic electrodynamics.
- **Electromagnetic Field Tensor:** The combination of the electric and magnetic field into a covariant tensor is called electromagnetic field tensor.

KEY FORMULAE

Spacetime Interval

$$(\Delta s)^2 = c^2t^2 - x^2 - y^2 - z^2$$

Rapidity

$$v = c\tanh\theta$$

Minkowski spacetime metric

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \text{diag}\,(-1,1,1,1)$$

Lorentz matrix

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} \Lambda^0_0 & \Lambda^0_1 & \Lambda^0_2 & \Lambda^0_3 \\ \Lambda^1_0 & \Lambda^1_1 & \Lambda^1_2 & \Lambda^1_3 \\ \Lambda^2_0 & \Lambda^2_1 & \Lambda^2_2 & \Lambda^2_3 \\ \Lambda^3_0 & \Lambda^3_1 & \Lambda^3_2 & \Lambda^3_3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Kronecker delta function

$$\delta^{\beta}_{\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Position Four Vector

$$x^{\mu} = \left(x^o, x^i\right) = \left(x^o, \mathbf{x}\right) = (ct, x, y, z)$$

Four vector derivative

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Velocity four vector

$$u^{\mu} = \frac{dx^{\mu}}{d\tau} = \gamma\frac{dx^{\mu}}{dt} = \gamma\,(c, \mathbf{v})$$

Acceleration four vector

$$a^{\mu} = \frac{du^{\mu}}{d\tau} = \gamma\frac{du^{\mu}}{dt} = \gamma^2\left(\gamma^2\frac{\mathbf{u}\cdot\mathbf{a}}{c}, \mathbf{a} + \gamma^2\frac{\mathbf{u}\cdot\mathbf{a}}{c^2}\mathbf{u}\right)$$

Momentum four vector

$$p^{\mu} = mu^{\mu} = m\gamma(c, \mathbf{v})$$

Energy-momentum relation

$$E^2 = p^2c^2 + m^2c^4$$

Force four vector

$$K^\mu = \frac{dp^\mu}{d\tau} = \gamma\frac{dp^\mu}{dt}$$

Four potential

$$A^\mu = \left(A^0, A^i\right) = \left(\frac{V}{c}, \mathbf{A}\right)$$

Covariant current density

$$J^\mu = \left(J^0, J^i\right) = (c\rho, \mathbf{J})$$

Electromagnetic field tensor

$$F_{\mu\nu} = \partial_\mu A_v - \partial_\nu A_\mu = \begin{pmatrix} F_{00} & F_{01} & F_{02} & F_{03} \\ F_{10} & F_{11} & F_{12} & F_{13} \\ F_{20} & F_{21} & F_{22} & F_{23} \\ F_{30} & F_{31} & F_{32} & F_{33} \end{pmatrix} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

Levi-Civita symbol

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if}ijk\text{is an even permutation of 123,} \\ -1 & \text{if}ijk\text{is an odd permutation of 123;} \\ 0 & \text{if}i = j = k \end{cases}$$

Lorentz force

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Covariant form of Maxwell's equation

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= -\frac{4\pi}{c}J^\nu \\ \partial_\mu \tilde{F}^{\mu\nu} &= 0 \end{aligned}$$

4.8.3 Problems

1. Two events occur at $(ct_1, x_1, y_1, z_1) = (3, 7, 0, 0)m$ and $(ct_2, x_2, y_2, z_2) = (5, 5, 0, 0)m$. What is their spacetime separation?

2. Show that in Minkowski space $v^\mu = (\gamma c, \gamma \vec{v})$, $p^\mu = \left(\frac{E}{c}, \vec{p}\right)$. By considering the invariant $p^\mu p_\mu$ deduce that $E^2 = c^2p^2 + m_o^2c^4$. Also show that the components of 4-force are $F^\mu = \left(\gamma \vec{v} \cdot \vec{F}, \gamma \vec{F}\right)$ and $\frac{dp^\mu}{d\tau} = \left(\frac{\gamma}{c}\frac{dE}{dt}, \gamma \frac{d\vec{p}}{dt}\right)$

3. Three events E_1, E_2 and E_3 are observed to occur at spacetime coordinates (x, t) of $(2, 3/c)$, $(4, 7/c)$ and $(6, 5/c)$ respectively, in same frame S. By considering the relevant spacetime intervals, establish which pairs of events could be casually connected.

4. Show that the line element of Minkowski space is invariant under Lorentz boost.

5. The Electric field E and the magnetic field B are related to the scalar potential ϕ and the vector potential **A** by $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \vec{\nabla} \times \vec{A}$. Define the four vector potential A^μ and the field strength tensor $F^{\mu\nu}$ and its dual $G^{\mu\nu}$. Write the components of $F^{\mu\nu}$ and its dual $G^{\mu\nu}$.

6. Prove that the first two equations of the special Lorentz transformation can be written in the form

$$ct' = -x \sinh\phi + ct \cosh\phi, x' = x \cosh\phi - ct \sinh\phi$$

where the rapidity ϕ is defined by $\phi = \tanh^{-1}\frac{v}{c}$. Establish also the following version of these equations:

$$ct' + x' = e^{-\phi}(ct + x), ct' - x' = e^{\phi}(ct - x), e^{2\phi} = \frac{(1 + v/c)}{(1 - v/c)}$$

What relation does ϕ have to θ, where θ is the angle of rotation of the spacetime axes.

Part II

General Relativity

Chapter 5

Elementary Tensors and Geometry

Learning Objectives

In this chapter you will learn

- About tensors
- Rank of tensor
- One forms and vectors
- Tensor field
- Parallel transport
- Metric tensor
- Christoffel symbols
- Riemann curvature tensor
- Ricci tensor
- Manifolds
- Line element and its types
- Geodesic
- Geodesic equation

KEY TERMS

tensor, rank of a tensor, one form, tensor field, parallel transport, metric tensor, Christoffel symbols, Riemann curvature tensor, Ricci tensor, manifold, line element, geodesic, affine parameter

5.1 Tensors

Q. What is tensor? Also discuss its ranks and its types.

A mathematical object analogous to vector represented by an array of components that are functions of the space coordinates is called tensor.

A tensor is a function of n vectors into real number. It is denoted by a^μ and given by

$$a^\mu = \left(a^1, a^2, \ldots, a^N\right)$$

where μ is an index having values $\mu = 1, 2, \ldots, N$ unless the contrary is specified like in the case of four vectors.

The choice of index as subscript or superscript is arbitrary. However a^μ(superscript) are vectors from a vector space and shows a column matrix while a_μ(superscript) are vectors from an orthogonal (dual) vector space and shows a row matrix.

5.1.1 Rank of Tensor

The number which shows the dimensions of tensor in array form is called rank of tensor.

Rank of tensor is also called order of tensor. Mathematically it shows the number of indices labelled on a tensor notation. It is also the sum of covariant and contravariant indices. But actually rank of a tensor gives the number of directions of a tensor. Rank also decides the type of tensor.

Rank Zero (Scalar)

Tensors of rank zero are called scalar.

A scalar quantity is invariant under rotation in any coordinate system. e.g., time.

Rank One (Vector)

Tensors of rank one are called vector.

There are three components of vector of Euclidean space and four components in Minkowski spacetime. e.g., velocity.

Rank Two (Tensor)

Rank of tensor two are called dyads.

This tensor maybe formed by combination of components of two vectors. There are three component of each vector then there maybe total number of $3 \times 3 = 9$ possible components of rank two tensor. e.g., metric tensor ($g_{\mu\nu}$), Kronecker delta ($\delta_\mu \nu$) etc.

Rank Three Tensor

This tensor maybe formed by combination of components of three vectors. There are three component of each vector then there maybe total number of $3 \times 3 \times 3 = 27$ possible components of rank three tensor. e.g., Levi-Civita symbol ($\varepsilon_{\mu\nu\lambda}$), Christoffel symbol ($\Gamma^\lambda_{\mu\nu}$) etc.

High Rank Tensor

The tensor of n rank is called high rank tensor.

e.g., Riemann curvature tensor $\left(R^\rho_{\mu\nu\lambda}\right)$

5.1.2 Number of Components

Number of components of a tensor of rank n is 3^n in Euclidean space and 4^n in Minkowski spacetime where 3 and 4 represents the dimensionality of Euclidean space and Minkowski space respectively.

Some examples to find the number of components along with tensor examples are given below in a table:

Rank	Geometry and Dimensions	Number of Components	Example
Vectors, $(n = 1)$	Euclidean space, $d = 3$	$3^1 = 3$	Momentum
Vectors, $(n = 1)$	Minkowski spacetime, $d = 4$	$4^1 = 4$	Four position
Dyads, $(n = 2)$	Minkowski spacetime, $d = 4$	$4^2 = 16$	Stress-energy tensor
Rank 4 $(n = 4)$	Minkowski spacetime, $d = 4$	$4^4 = 256$	Riemann Curvature tensor

5.1.3 One Forms and Vectors

A covariant tensor of rank 1 is called one-form.

It is also called covector or covariant vector. By convention, index of a one-form is written as subscript while that of a vector is written as superscript. As one-forms are from dual vector space that's why it is also called dual vector. The set of basis of one-form is a vector space. One-form basis in spacetime is called tetrad and given by

$$e_\mu = (e_t, e_i) = (t, \hat{x}, \hat{y}, \hat{z})$$

The metric tensor of tetrad is given by

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

5.1.4 Tensor Field

A function of points in Euclidean space or manifold that defines the value of a field using tensors is called tensor field.

Tensor field is a general form of field just like a tensor is a generalization of scalars. e.g., Riemann curvature tensor $\left(R^{\rho}_{\mu\nu\lambda}\right)$, stress energy tensor $(T^{\mu\nu})$ and electromagnetic field tensor . In a field to every point there is a corresponding tensor that is basically tensor field.

5.1.5 Parallel Transport

Q. What is parallel transport? Elaborate with diagram.

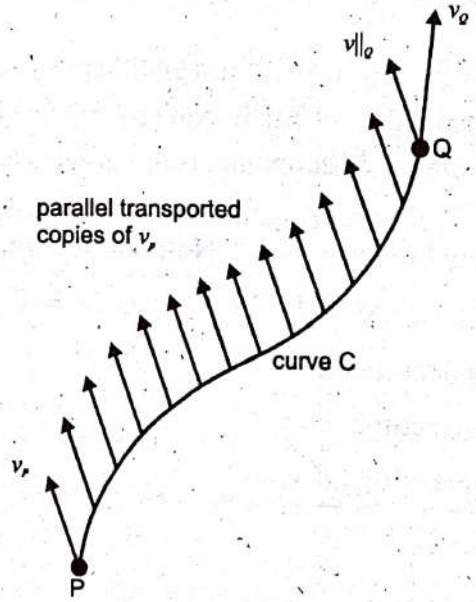


Figure 5.1: *The parallel transport of a vector along a curve from P to Q*

The method of transporting geometrical data along a curve in manifold preserving the direction of vectors of the manifold with respect to connection is called parallel transport.

Connection is a notion used to relate two vectors of the manifold. It also provides a way of connecting the geometries of different points on a manifold. Parallel transport is a fundamental concept in differential geometry. It envails the importance of covariant differentiation and Christoffel symbols. A curve can be drawn by the parallel transport of its tangent vector. The length of the curve can be measured using parallel transport of a vector.(Fig.5.1)

For instance, the speed of the wind in the atmosphere is to be compared at two different points P and Q. To compare the velocity observer at point Q must have information about velocity at point P. Observer at point P sends a copy of velocity vector $v_{\|Q}$ towards point Q. this was sent by a parallel transport of the v_p. This can be easily done using Cartesian coordinates. It is not simple in spherical or other coordinates because in Cartesian coordinates system basis vectors remain same but in spherical or other coordinates systems basis vectors transform too.

5.2 Covariant Metric Tensor

A covariant symmetric tensor of rank 2 that defines the mapping of scalar product between two vectors is called covariant metric tensor.

It is denoted by $g_{\mu\nu}$. The metric tensor is function of position. The metric tensor is symmetric so

$$g_{\mu\nu} = g_{\nu\mu}$$

The inverse of metric is written in contravariant form such that

$$g_{\mu\lambda}g^{\lambda\nu} = \delta^{\nu}_{\mu}$$

Where $g^{\lambda\nu}$ is called dual tensor or inverse metric tensor. Since metric tensor may depend on position but is a mixed metric tensor which does not depends on position and called **Kronecker delta**. It is related with Lorentz transformation as

$$\eta_{\mu\nu} = \left(\Lambda^{\alpha}_{\mu}\right)^{T} \eta_{\alpha\beta}\Lambda^{\beta}_{\nu}$$

Metric tensor also provides mapping between covariant and contravariant vectors. In other words, it can also be used to convert a contravariant vector into covariant vector as

$$a_{\mu} = g_{\mu\nu}a^{\nu}$$

Similarly inverse metric tensor can be used to convert a covariant vector into contravariant vector as

$$a^{\mu} = g^{\mu\nu}a_{\nu}$$

Metric tensor is also used to form a scalar from any second rank general tensor ($T_{\mu\nu}$) as

$$g^{\mu\nu}T_{\mu\nu} = g_{\mu\nu}T^{\mu\nu} = g^{\nu}_{\mu}T^{\mu}_{\nu} = \delta^{\nu}_{\mu}T^{\mu}_{\nu} = T^{\mu}_{\mu}$$

5.3 Covariant Differentiation

Q. What is covariant differentiation? Describe in details.

A way of specifying a derivative along tangent vector of a manifold is called covariant differentiation.

Consider a vector **a** (n components) such that

$$\mathbf{a} = a^{\mu} e_{\mu}$$

Where a^{μ} are components of vector and e_{μ} are basis vectors.

Its covariant derivative will be

$$\begin{aligned}\frac{\partial \mathbf{a}}{\partial x^{\nu}} &= \frac{\partial}{\partial x^{\nu}}\left(a^{\mu} e_{\mu}\right)\\ \frac{\partial \mathbf{a}}{\partial x^{\nu}} &= \frac{\partial a^{\mu}}{\partial x^{\nu}} e_{\mu} + a^{\mu}\frac{\partial e_{\mu}}{\partial x^{\nu}}\end{aligned} \tag{5.1}$$

For Cartesian coordinates the vector **a** is

$$\mathbf{a} = a_x e_x + a_y e_y + a_z e_z$$

and its covariant derivative along x-axis will be

$$\frac{\partial \mathbf{a}}{\partial x} = \frac{\partial a_x}{\partial x} e_x + \frac{\partial a_y}{\partial x} e_y + \frac{\partial a_y}{\partial x} e_y$$

Since second term of Eq.(5.1) is vanished but it is not true in general.

For spherical coordinates (r, θ, ϕ)

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi\\ y &= r\sin\theta\sin\phi\\ z &= r\cos\theta\end{aligned}\right\}$$

The relationship between two coordinates systems can also be determined by a matrix Λ given by

$$\Lambda = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta\\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0\end{pmatrix}$$

The basis vectors in spherical coordinates are

$$\begin{aligned} e_r &= \sin\theta\cos\phi e_x + \sin\theta\sin\phi e_y + \cos\theta e_z\\ e_{\theta} &= r\cos\theta\cos\phi e_x + r\cos\theta\sin\phi e_y - r\sin\theta e_z\\ e_{\phi} &= -r\sin\theta\sin\phi e_x + r\sin\theta\cos\phi e_y + (0)\, e_z\end{aligned}$$

Differentiating e_r with respect to (r, θ, ϕ)

$$\begin{aligned}
\frac{\partial e_r}{\partial r} &= \frac{\partial}{\partial r}\left(\sin\theta\cos\phi e_x + \sin\theta\sin\phi e_y + \cos\theta e_z\right) \\
\frac{\partial e_r}{\partial r} &= 0 \\
\frac{\partial e_r}{\partial \theta} &= \frac{\partial}{\partial \theta}\left(\sin\theta\cos\phi e_x + \sin\theta\sin\phi e_y + \cos\theta e_z\right) \\
&= \cos\theta\cos\phi e_x + \cos\theta\sin\phi e_y - \sin\theta e_z \\
\frac{\partial e_r}{\partial \theta} &= \frac{1}{r} e_\theta \neq 0 \\
\frac{\partial e_r}{\partial \phi} &= \frac{\partial}{\partial \phi}\left(\sin\theta\cos\phi e_x + \sin\theta\sin\phi e_y + \cos\theta e_z\right) \\
&= -\sin\theta\sin\phi e_x + \sin\phi\cos\phi e_y + (0)\, e_z \\
\frac{\partial e_r}{\partial \phi} &= \frac{1}{r} e_\phi \neq 0
\end{aligned}$$

Again the same case. Thus in Eq.(5.1) second term (derivative of basis vector) does not vanish in general. Hence Eq.(5.1) gives derivative of a basis vector.

5.4 Christoffel Symbols

Q. What are Christoffel symbols? Also make a relation between Christofell symbols and metric tensor.

An array of numbers describing a metric connection is called christofell symbol.

This is λth component of derivative of a basis vector. It gives the magnitude of one component of derivative of a vector.

Let a vector a (a^1, a^2, a^3) in Euclidean space in its component form is

$$\mathbf{a} = a_x e_x + a_y e_y + a_z e_z$$

Where (e_x, e_y, e_z) are basis vectors along $x-$, $y-$ and $z-$axes. Its derivative in a particular direction can be written as

$$\frac{\partial \mathbf{a}}{\partial x^j} = \sum_i \left(\frac{\partial a_i}{\partial x^j} e_i + a_i \frac{\partial e_i}{\partial x^j}\right)$$

Where $x^j = (x, y, z)$.

In Minkowski space, four vector is

$$a^\mu = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$$

And its derivative in a particular direction is

$$\frac{\partial a^{\mu}}{\partial x^{\nu}} = \sum_{\mu}\left(\frac{\partial a_{\mu}}{\partial x^{\nu}}e_{\mu} + a_{\mu}\frac{\partial e_{\mu}}{\partial x^{\nu}}\right)$$

As $\frac{\partial e_{\mu}}{\partial x^{\nu}}$ is a vector itself. In Riemannian space it can be written as a linear combination of basis vector as

$$\frac{\partial e_{\mu}}{\partial x^{\nu}} = \sum_{\lambda}\Gamma^{\lambda}_{\mu\nu}e_{\lambda} \tag{5.2}$$

Where $\Gamma^{\lambda}_{\mu\nu}$ is λth component of $\frac{\partial e_{\mu}}{\partial x^{\nu}}$ called Christofell symbol. It is a function of coordinates. These are also called affine connection. The index μ represents the basis vector being differentiated, ν represents the coordinate with respect to which it is being differentiated and λ represents the component of resulting vector.

In spherical coordinates basis vectors are

$$\begin{aligned} e_r &= \sin\theta\cos\phi e_x + \sin\theta\sin\phi e_y + \cos\theta e_z \\ e_\theta &= r\cos\theta\cos\phi e_x + r\cos\theta\sin\phi e_y - r\sin\theta e_z \\ e_\phi &= -r\sin\theta\sin\phi e_x + r\sin\theta\cos\phi e_y + (0)\, e_z \end{aligned}$$

Christofell symbols are also tensor. When a tensor of rank (m, n) is differentiated it should give a tensor of rank (m, n_1). But in case of Cartesian coordinates it is not true. So a correction term in needed. The christofell symbols play the role of correction term. Another reason to introduce christofell symbol is that partial derivative of tensor is not a tensor.

In matrix form for spherical coordinates Christofell symbols are given as

$$\Gamma^{r} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -rsin^2\theta \end{pmatrix}, \quad \Gamma^{\phi} = \begin{pmatrix} 0 & \frac{1}{r} & 0 \\ \frac{1}{r} & 0 & 0 \\ 0 & 0 & \sin\theta\cos\theta \end{pmatrix}, \quad \Gamma^{\phi} = \begin{pmatrix} 0 & 0 & \frac{1}{r} \\ 0 & 0 & \cot\theta \\ \frac{1}{r} & \cot\theta & 0 \end{pmatrix}$$

5.4.1 Christofell Symbols and Metric Tensor

Christofell symbols are related with basis vector as

$$\Gamma^{\lambda}_{\mu\nu}e_{\lambda} = \frac{\partial e_{\mu}}{\partial x^{\nu}}$$

Taking dot product with a basis vector

$$\Gamma^{\lambda}_{\mu\nu}e_{\lambda}\cdot e_{\rho} = \frac{\partial e_{\mu}}{\partial x^{\nu}}\cdot e_{\rho}$$

As $e_\lambda \cdot e_\rho = \delta^\rho_\lambda$, so

$$\begin{aligned}
\Gamma^\lambda_{\mu\nu}\delta^\rho_\lambda &= \frac{\partial e_\mu}{\partial x^\nu} \cdot e_\rho \\
\Gamma^\rho_{\mu\nu} &= \frac{\partial e_\mu}{\partial x^\nu} \cdot e_\rho \\
\Gamma^\rho_{\mu\nu} &= \frac{1}{2}\left(2\frac{\partial e_\mu}{\partial x^\nu} \cdot e_\rho\right)
\end{aligned}$$

As $\frac{\partial e_\mu}{\partial x^\nu} = \frac{\partial e_\nu}{\partial x^\mu}$, so

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}\frac{\partial e_\mu}{\partial x^\nu} \cdot e_\rho + \frac{1}{2}\frac{\partial e_\nu}{\partial x^\mu} \cdot e_\rho$$

Adding and subtracting the same term

$$\begin{aligned}
\Gamma^\rho_{\mu\nu} &= \frac{1}{2}\frac{\partial e_\mu}{\partial x^\nu} \cdot e_\rho + \left(\frac{1}{2}g^{\lambda\rho}\frac{\partial e_\lambda}{\partial x^\nu} \cdot e_\mu - \frac{1}{2}g^{\lambda\rho}\frac{\partial e_\nu}{\partial x^\lambda} \cdot e_\mu\right) \\
&\quad + \frac{1}{2}\frac{\partial e_\nu}{\partial x^\mu} \cdot e_\rho + \left(\frac{1}{2}g^{\lambda\rho}\frac{\partial e_\lambda}{\partial x^\mu} \cdot e_\nu - \frac{1}{2}g^{\lambda\rho}\frac{\partial e_\mu}{\partial x^\lambda} \cdot e_\nu\right) \\
\Gamma^\rho_{\mu\nu} &= \frac{1}{2}g^{\lambda\rho}\frac{\partial e_\mu}{\partial x^\nu} \cdot e_\lambda + \left(\frac{1}{2}g^{\lambda\rho}\frac{\partial e_\lambda}{\partial x^\nu} \cdot e_\mu - \frac{1}{2}g^{\lambda\rho}\frac{\partial e_\nu}{\partial x^\lambda} \cdot e_\mu\right) \\
&\quad + \frac{1}{2}g^{\lambda\rho}\frac{\partial e_\nu}{\partial x^\mu} \cdot e_\lambda + \left(\frac{1}{2}g^{\lambda\rho}\frac{\partial e_\lambda}{\partial x^\mu} \cdot e_\nu - \frac{1}{2}g^{\lambda\rho}\frac{\partial e_\mu}{\partial x^\lambda} \cdot e_\nu\right) \\
\Gamma^\rho_{\mu\nu} &= \frac{1}{2}g^{\lambda\rho}\left[\left(\frac{\partial e_\mu}{\partial x^\nu} \cdot e_\lambda + \frac{\partial e_\lambda}{\partial x^\nu} \cdot e_\mu\right) + \left(\frac{\partial e_\nu}{\partial x^\mu} \cdot e_\lambda + \frac{\partial e_\lambda}{\partial x^\mu} \cdot e_\nu\right) - \left(\frac{\partial e_\nu}{\partial x^\lambda} \cdot e_\mu + \frac{\partial e_\mu}{\partial x^\lambda} \cdot e_\nu\right)\right] \\
\Gamma^\rho_{\mu\nu} &= \frac{1}{2}g^{\lambda\rho}\left[\frac{\partial}{\partial x^\nu}(e_\lambda \cdot e_\mu) + \frac{\partial}{\partial x^\mu}(e_\nu \cdot e_\lambda) - \frac{\partial}{\partial x^\lambda}(e_\mu \cdot e_\nu)\right] \\
\Gamma^\rho_{\mu\nu} &= \frac{1}{2}g^{\lambda\rho}\left(\frac{\partial g_{\lambda\mu}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right) \\
\Gamma^\rho_{\mu\nu} &= \frac{1}{2}g^{\lambda\rho}\left(\partial_\nu g_{\lambda\mu} + \partial_\mu g_{\nu\lambda} - \partial_\lambda g_{\mu\nu}\right)
\end{aligned}$$

Once metric tensor is known the Christofell symbols can be determined in any coordinates system.

$$\Gamma^\beta_{\mu\nu} = \frac{\partial x^\beta}{\partial q^\alpha}\frac{\partial^2 q^\alpha}{\partial x^\mu \partial x^\nu}$$

5.5 Riemann Curvature Tensor

Q. What is Riemann curvature tensor. Derive its formula. Also develop Bianchi identity.

A tensor that defines the tensor filed at each point in Riemannian manifold is called Riemannian tensor.

It is used to express the curvature of Riemannian manifold. That's why it is also called Riemannian curvature tensor.

It is given by

$$R^{\rho}_{\mu\nu\lambda} = \partial_{\nu}\Gamma^{\rho}_{\mu\lambda} - \partial_{\lambda}\Gamma^{\rho}_{\mu\nu} + \Gamma^{\sigma}_{\mu\lambda}\Gamma^{\rho}_{\sigma\nu} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\sigma\lambda}$$

Proof

Consider the parallel transport of a vector in a parallelogram $ABCD$. The sides of parallelogram are a^{ν} and b^{μ} and their respective opposite sides are $(a+da)^{\nu}$ and $(b+db)^{\mu}$ respectively. Then using parallel transport relations

$$\left.\begin{aligned} (a+da)^{\nu} &= a^{\nu} - \Gamma^{\nu}_{\rho\sigma}a^{\rho}b^{\sigma} \\ (b+db)^{\mu} &= b^{\mu} - \Gamma^{\mu}_{\rho\sigma}a^{\rho}b^{\sigma} \end{aligned}\right\} \tag{5.3}$$

From fig.(—) parallel transport ABC gives

$$dA^{\rho}_{ABC} = dA^{\rho}_{AB} + dA^{\rho}_{BC}$$

As $\dfrac{dA^{\mu}}{dx^{\nu}} = -\Gamma^{\mu}_{\lambda\nu}A^{\lambda}$, so

$$dA^{\rho}_{ABC} = -(\Gamma^{\rho}_{\sigma\nu}A^{\sigma})_{A}a^{\nu} - (\Gamma^{\rho}_{\sigma\lambda}A^{\sigma})_{B}(b+db)^{\lambda} \tag{5.4}$$

Subscripts A and B gives the respective positions where these functions are to be evaluated. Now

$$\begin{aligned} (\Gamma^{\rho}_{\sigma\lambda})_{B} &= (\Gamma^{\rho}_{\sigma\lambda})_{A} + a^{\nu}(\partial_{\nu}\Gamma^{\rho}_{\sigma\lambda})_{A} \\ (A^{\sigma})_{B} &= (A^{\sigma})_{A} + a^{\nu}(\partial_{\nu}A^{\sigma})_{A} \end{aligned}$$

As $\dfrac{dA^{\mu}}{dx^{\nu}} = -\Gamma^{\mu}_{\lambda\nu}A^{\lambda}$, so

$$(A^{\sigma})_{B} = (A^{\sigma})_{A} + a^{\nu}\left(\Gamma^{\sigma}_{\mu\nu}A^{\mu}\right)_{A} \tag{5.5}$$

Putting Eq.(5.3) and (5.5) in Eq.(5.4)

$$\begin{aligned} dA^{\rho}_{ABC} &= -\Gamma^{\rho}_{\sigma\nu}A^{\sigma}a^{\nu} - (\Gamma^{\rho}_{\sigma\lambda} + a^{\nu}\partial_{\nu}\Gamma^{\rho}_{\sigma\lambda})\left(A^{\sigma} + a^{\nu}\Gamma^{\sigma}_{\mu\nu}A^{\mu}\right)\left(b^{\mu} - \Gamma^{\mu}_{\rho\sigma}a^{\rho}b^{\sigma}\right) \\ dA^{\rho}_{ABC} &= -\Gamma^{\rho}_{\sigma\nu}A^{\sigma}a^{\nu} - (\Gamma^{\rho}_{\sigma\lambda} + a^{\nu}\partial_{\nu}\Gamma^{\rho}_{\sigma\lambda})\left(A^{\sigma}b^{\mu} - A^{\sigma}\Gamma^{\mu}_{\rho\sigma}a^{\rho}b^{\sigma} + a^{\nu}\Gamma^{\sigma}_{\mu\nu}A^{\mu}b^{\mu} - a^{\nu}\Gamma^{\sigma}_{\mu\nu}A^{\mu}\Gamma^{\mu}_{\rho\sigma}a^{\rho} \right. \end{aligned}$$

Multiplying and keeping terms $\mathcal{O}\left(ab\right)$

$$dA^{\rho}_{ABC} = -\Gamma^{\rho}_{\sigma\nu}A^{\sigma}a^{\nu} - \Gamma^{\rho}_{\sigma\lambda}A^{\sigma}b^{\mu} + A^{\sigma}\Gamma^{\rho}_{\sigma\lambda}\Gamma^{\mu}_{\rho\sigma}a^{\rho}b^{\sigma} - a^{\nu}\partial_{\nu}\Gamma^{\rho}_{\sigma\lambda}\Gamma^{\mu}_{\rho\sigma}a^{\rho}b^{\sigma} - \Gamma^{\rho}_{\sigma\lambda}a^{\nu}\Gamma^{\sigma}_{\mu\nu}A^{\mu}b^{\lambda} \tag{5.6}$$

Similarly from CDA, the parallel transport is

$$dA^{\rho}_{CDA} = -\Gamma^{\rho}_{\sigma\nu}A^{\sigma}b^{\nu} + \Gamma^{\rho}_{\sigma\lambda}A^{\sigma}a^{\mu} - A^{\sigma}\Gamma^{\rho}_{\sigma\lambda}\Gamma^{\mu}_{\rho\sigma}b^{\rho}a^{\sigma} - b^{\nu}\partial_{\nu}\Gamma^{\rho}_{\sigma\lambda}A^{\sigma}a^{\mu} - \Gamma^{\rho}_{\sigma\lambda}b^{\nu}\Gamma^{\sigma}_{\mu\nu}A^{\mu}a^{\lambda} \tag{5.7}$$

Subtracting Eq.(5.7) from (5.6)

$$\begin{aligned} dA^{\rho} &= -\left(\partial_{\nu}\Gamma^{\rho}_{\mu\lambda} - \partial_{\lambda}\Gamma^{\rho}_{\mu\nu} + \Gamma^{\rho}_{\sigma\nu}\Gamma^{\sigma}_{\mu\lambda} - \Gamma^{\rho}_{\sigma\lambda}\Gamma^{\sigma}_{\mu\nu}\right)A^{\mu}a^{\nu}b^{\lambda} \\ dA^{\rho} &= -R^{\rho}_{\mu\nu\lambda}A^{\mu}a^{\nu}b^{\lambda} \end{aligned}$$

Where $R^{\rho}_{\mu\nu\lambda} = \partial_\nu \Gamma^{\rho}_{\mu\lambda} - \partial_\lambda \Gamma^{\rho}_{\mu\nu} + \Gamma^{\rho}_{\sigma\nu}\Gamma^{\sigma}_{\mu\lambda} - \Gamma^{\rho}_{\sigma\lambda}\Gamma^{\sigma}_{\mu\nu}$

This equation gives Riemann tensor which describes the intrinsic geometry of space. Using metric tensor above equation becomes

$$\begin{aligned}
R_{\mu\nu\lambda\rho} &= g_{\rho\sigma} R^{\rho}_{\mu\nu\lambda} \\
&= g_{\rho\sigma}\left(\partial_\nu \Gamma^{\sigma}_{\mu\lambda} - \partial_\lambda \Gamma^{\sigma}_{\mu\nu} + \Gamma^{\sigma}_{\tau\nu}\Gamma^{\sigma}_{\mu\lambda} - \Gamma^{\sigma}_{\tau\lambda}\Gamma^{\tau}_{\mu\nu}\right) \\
R_{\mu\nu\lambda\rho} &= \partial_\nu \Gamma_{\mu\lambda\rho} - \partial_\lambda \Gamma_{\mu\nu\rho} + \Gamma_{\tau\nu\rho}\Gamma^{\tau}_{\mu\lambda} - \Gamma_{\rho\tau\lambda}\Gamma^{\tau}_{\mu\nu}
\end{aligned}$$

As $\Gamma_{\mu\nu\lambda} = \dfrac{1}{2}\left(\dfrac{\partial g_{\nu\lambda}}{\partial x^\mu} + \dfrac{\partial g_{\lambda\mu}}{\partial x^\nu} - \dfrac{\partial g_{\mu\nu}}{\partial x^\lambda}\right)$, so

$$\begin{aligned}
R_{\mu\nu\lambda\rho} &= \frac{1}{2}\partial_\nu\left(\frac{\partial g_{\lambda\rho}}{\partial x^\mu} + \frac{\partial g_{\rho\mu}}{\partial x^\lambda} - \frac{\partial g_{\mu\lambda}}{\partial x^\rho}\right) - \frac{1}{2}\partial_\lambda\left(\frac{\partial g_{\nu\rho}}{\partial x^\mu} + \frac{\partial g_{\rho\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right) + \Gamma_{\tau\nu\rho}\Gamma^{\tau}_{\mu\lambda} - \Gamma_{\rho\tau\lambda}\Gamma^{\tau}_{\mu\nu} \\
R_{\mu\nu\lambda\rho} &= \frac{1}{2}\left(\frac{\partial^2 g_{\lambda\rho}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\nu}}{\partial x^\lambda \partial x^\rho} - \frac{\partial^2 g_{\mu\lambda}}{\partial x^\nu \partial x^\rho} - \frac{\partial^2 g_{\nu\rho}}{\partial x^\lambda \partial x^\mu}\right) + \Gamma_{\tau\nu\rho}\Gamma^{\tau}_{\mu\lambda} - \Gamma_{\rho\tau\lambda}\Gamma^{\tau}_{\mu\nu} \\
R_{\mu\nu\lambda\rho} &= \frac{1}{2}\left(\partial_\nu\partial_\mu g_{\lambda\rho} + \partial_\lambda\partial_\rho g_{\mu\nu} - \partial_\nu\partial_\rho g_{\mu\lambda} - \partial_\lambda\partial_\mu g_{\nu\rho}\right) + \Gamma_{\tau\nu\rho}\Gamma^{\tau}_{\mu\lambda} - \Gamma_{\rho\tau\lambda}\Gamma^{\tau}_{\mu\nu}
\end{aligned}$$

For flat manifold $R_{\mu\nu\lambda\rho} = 0$

The symmetric and antisymmetric properties of Riemann tensor are

$$R_{\mu\nu\lambda\rho} = R_{\lambda\rho\mu\nu} = -R_{\mu\nu\rho\lambda} = -R_{\nu\mu\lambda\rho}$$

There are total n^4 components of Riemann tensor in n-dimensional space. Using above antisymmetric property only $\dfrac{n^2\left(n^2-1\right)}{12}$ independent components left. For spacetime $n = 4$ and 20 components.

By cyclic permutation of indices the sum of three Riemann tensors will be zero. i.e.,

$$R_{\mu\nu\lambda\rho} + R_{\mu\rho\nu\lambda} + R_{\mu\lambda\rho\nu} = 0$$

Riemann tensor also satisfies the **Bianchi identity**

$$\nabla_\mu R_{\rho\sigma\nu\lambda} + \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\nu R_{\rho\sigma\lambda\mu} = 0$$

5.6 Ricci Tensor and Scalar

What is Ricci tensor and scalar. Find their relation.

A tensor which is used to define the part of the curvature of spacetime in Einstein's tensor is called Ricci tensor.

Ricci tensor can be determined from the Riemann tensor as

$$R_{\mu\nu} = R^{\lambda}_{\mu\nu\lambda} \tag{5.8}$$

Geometrically, the Ricci tensor is the mathematical object that controls the growth rate of the volume of metric balls in a manifold.

The Ricci tensor is symmetric. i.e.,

$$R_{\mu\nu} = R_{\nu\mu}$$

The Ricci scalar can also be determined from Ricci tensor as

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} g^{\lambda\rho} R_{\mu\nu\lambda\rho} = R^{\mu}_{\mu}$$

Using Ricci tensor and Ricci scalar, Einstein's field tensor can also be determined.

5.7 Einstein's Tensor

What is Einstein's tensor? Find its formula.

A tensor used to express the curvature of pseudo Riemannian manifold in differential geometry is called Einstein's tensor.

It is given by

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$$

Where $R^{\mu\nu}$ is Ricci tensor and R is Ricci scalar.

As $R^{\mu\nu}$ and $g^{\mu\nu}$ are symmetric so $G^{\mu\nu}$ is also symmetric. i.e.,

$$G^{\mu\nu} = G^{\nu\mu}$$

Since it is a rank 2 tensor so there are total 16 components containing 10 independent components.

Its divergence is zero. i.e.,

$$\nabla_{\mu} G^{\mu\nu} = 0$$

5.8 Riemannian Geometry

Q. What is Riemannian geometry? Also describe manifold.

The branch of mathematics which deals with the study of Riemannian manifolds and smooth manifolds with a Riemannian metric is called Riemannian geometry.

Riemannian geometry is basically study of curved surfaces and spaces. This geometry is generalization of the differential geometry in three dimensions. It gives the concept of angle, length of curves, surface area and volume.

5.8.1 Manifolds

A continuous space of points that resembles with Euclidean space near each point and maybe curved globally and flat locally is called manifold.

Its general form is n-manifold in n dimensions.

- One dimensional manifolds include lines and circles.
- Two dimensional manifolds are also called surfaces and includes plane and sphere.

Manifold describe complicated structures in a simple way.

Manifold is flat locally and curved globally. For example earth is a curved surface globally but it seems to be flat locally.

5.8.2 Riemann Manifold

A real manifold M equipped with an inner product g_ρ on the tangent space $T_\rho M$ at each point ρ that varies smoothly from point to point in the sense that if X and Y are differentiable vector fields on M, then $\rho \to g_\rho \left(X|_\rho, Y|_\rho\right)$ is a smooth function. The family g_ρ of inner products is called a Riemannian metric.

5.8.3 Riemannian Geometry and Theory of Relativity

In 1905, Einstein developed special theory of relativity but this theory is not consistent with Newton's law of gravitation. In 1907, Einstein started work on general relative motion that resulted in the development of general theory of relativity. When Minkowski revealed the geometric aspect of special theory of relativity, Einstein realised the importance of geometry in the development of general theory of relativity. The geometry later he used for general theory of relativity is Riemannian geometry.

5.9 Line Element

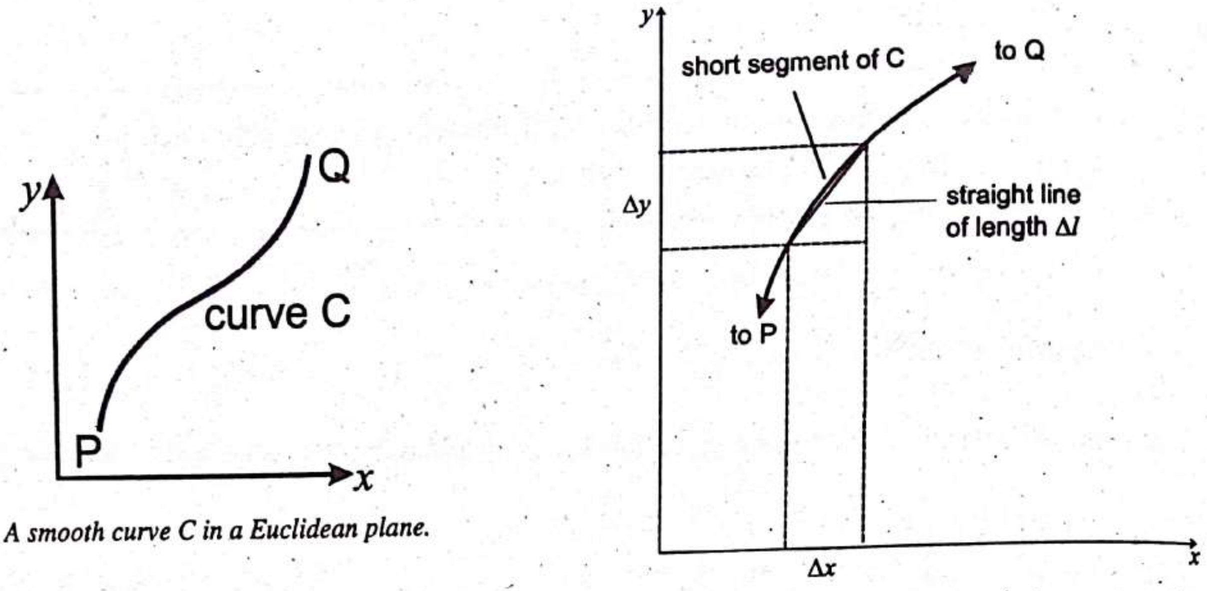
Q. What is line element? Form its formula in different coordinate systems.

5.9.1 Line Element on a Surface

A line segment associated with an infinitesimal displacement vector in a metric space is called line element.

In Euclidean plane the separation between two points $P\left(x_1, y_1\right)$ and $Q\left(x_2, y_2\right)$ of a curve C (Fig.5.2a) is determined by simple distance formula given by

$$\ell = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$$



(a) *A smooth curve C in a Euclidean plane.*

(b) *Each short segment of a curve C can be approximated by a straight line of length* $\Delta\ell$

Figure 5.2: *Line element on a curve*

A more accurate length can be determined by dividing whole length into n small segments of lengths $\Delta\ell_i (i = 1, 2, \ldots, n)$ such that each segment is a straight line. (Fig.5.2b). Then the total length of the curve will be

$$\ell = \Delta\ell_1 + \Delta\ell_2 + \ldots + \Delta\ell_n = \sum_{i=1}^{n} \Delta\ell_i$$

An even more accurate length can be determined by increasing number of segments. i.e., taking the limit $\Delta\ell \to 0$. Thus the length of the curve C will be

$$\ell = \int_{(x_1,y_1)}^{(x_2,y_2)} d\ell$$

Where $d\ell$ is the line element given by

$$d\ell = \sqrt{dx^2 + dy^2} \tag{5.9}$$

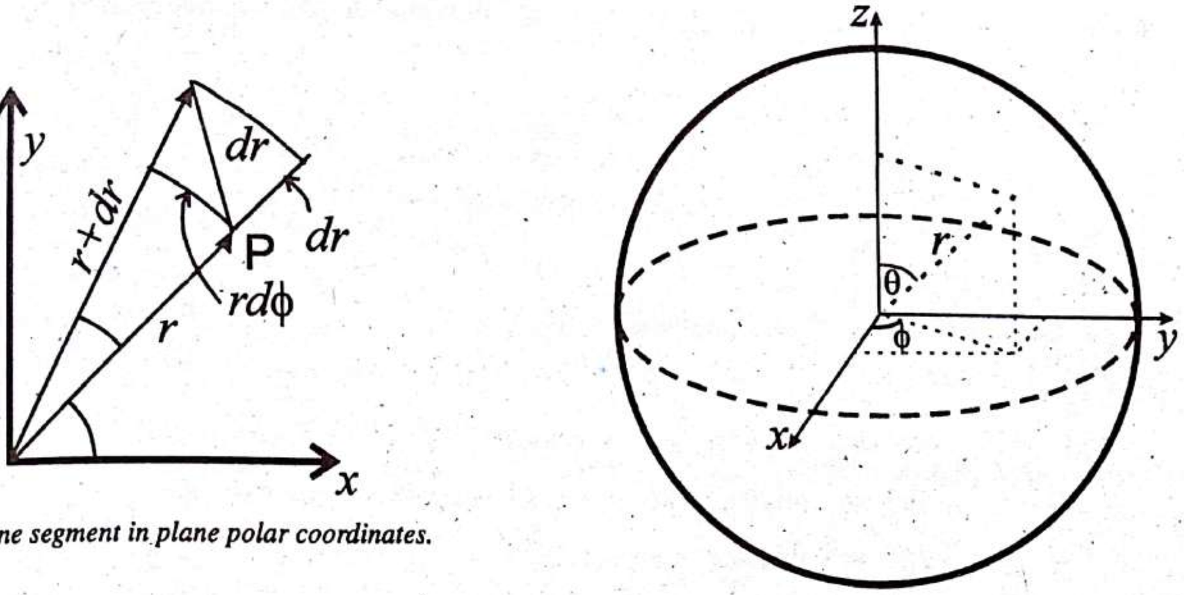
The other way to represent the curve is using a independent parameter (variable) α. The coordinates of the plane are expressed as $x(\alpha)$ and $y(\alpha)$ using this parameter. Some of the examples of parametric representation of curve are:

- A circle $x^2 + y^2 = r^2$ with $x = r\cos\theta$ and $y = r\sin\theta$ where θ is a parameter.
- A parabola $y = x^2$ with $x = \alpha$ and $y = \alpha^2$

Similarly interms of a parameter α the total length ℓ can be written as

$$\ell = \int_{(x_1,y_1)}^{(x_2,y_2)} d\ell = \int \sqrt{\left(\frac{dx}{d\alpha}\right)^2 + \left(\frac{dy}{d\alpha}\right)} d\alpha$$

The length of the curve can also be determined using plane polar coordinates (r, θ)(Fig.5.3a) given by



(a) *A line segment in plane polar coordinates.*

(b) *Spherical coordinates*

Figure 5.3: *Line element on a curve in plane polar and spherical coordinates*

$$x = r\cos\theta, \qquad y = r\sin\theta$$

Taking differentials

$$dx = -r\sin\theta d\theta + \cos\theta dr, \qquad dy = r\cos\theta d\theta + \sin\theta dr$$

Putting values in Eq.(5.9)

$$\begin{aligned} d\ell &= \sqrt{(-r\sin\theta d\theta + \cos\theta dr)^2 + (r\cos\theta d\theta + \sin\theta dr)^2} \\ &= \sqrt{r^2\sin^2\theta + \cos^2\theta dr^2 - 2r\sin\theta\cos\theta drd\theta + r^2\cos^2\theta d\theta^2 + \sin^2\theta dr^2 + 2r\sin\theta\cos\theta drd\theta} \\ &= \sqrt{r^2\left(\sin^2\theta + \cos^2\theta\right) + \left(\sin^2\theta + \cos^2\theta\right)dr^2} \\ d\ell &= \sqrt{r^2 d\theta^2 + dr^2} \end{aligned}$$

The total length will be

$$\ell = \int d\ell = \int \sqrt{r^2 d\theta^2 + dr^2}$$

5.9.2 Line Element in Space

The line element in space using Cartesian coordinates becomes

$$\ell = \int d\ell = \int \sqrt{dx^2 + dy^2 + dz^2} \tag{5.10}$$

In three dimensional Euclidean space the line element can be written as

$$d\ell = \sqrt{dx^2 + dy^2 + dz^2}$$

The line element can also be written using spherical coordinates. As in spherical coordinates (r,θ,ϕ) (Fig.5.3b)

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

Taking differentials

$$\begin{aligned} dx &= \sin\theta\cos\phi dr + r\cos\theta\cos\phi d\theta - r\sin\theta\sin\phi d\phi \\ dy &= \sin\theta\sin\phi dr + r\cos\theta\sin\phi d\theta + r\sin\theta\cos\phi d\phi \\ dz &= \cos\theta dr - r\sin\theta d\theta \end{aligned}$$

Squaring and adding above three equations

$$\begin{aligned} dx^2+dy^2+dz^2 &= (\sin\theta\cos\phi dr + r\cos\theta\cos\phi d\theta - r\sin\theta\sin\phi d\phi)^2 \\ &+ (\sin\theta\sin\phi dr + r\cos\theta\sin\phi d\theta + r\sin\theta\cos\phi d\phi)^2 + (\cos\theta dr - r\sin\theta d\theta)^2 \\ &= \sin^2\theta\left(\sin^2\phi+\cos^2\phi\right)dr^2 + r^2\cos^2\theta\left(\sin^2\phi+\cos^2\phi\right)d\theta^2 \\ &+ r^2\sin^2\theta\left(\sin^2\phi+\cos^2\phi\right)d\phi^2 + 2r\sin\theta\cos\theta\left(\sin^2\phi+\cos^2\phi\right)drd\theta \\ &+ \cos^2\theta dr^2 + r^2\sin^2\theta d\theta^2 - 2r\sin\theta\cos\theta drd\theta \\ &= \sin^2\theta dr^2 + r^2\cos^2\theta d\theta^2 + r^2\sin^2\theta d\phi^2 + 2r\sin\theta\cos\theta drd\theta \\ &+ \cos^2\theta dr^2 + r^2\sin^2\theta d\theta^2 - 2r\sin\theta\cos\theta drd\theta \\ dx^2+dy^2+dz^2 &= dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \end{aligned}$$

Putting value in Eq.(5.10)

$$\ell = \int\sqrt{dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2}$$

For constant $r=R$, $dr=0$ so that

$$\begin{aligned} \ell &= \int\sqrt{(0)^2+R^2d\theta^2+R^2\sin^2\theta d\phi^2} \\ \ell &= R\int\sqrt{d\theta^2+\sin^2\theta d\phi^2} \end{aligned}$$

This is line element on the surface of the sphere. Sphere itself is a three dimensional shape but in above expression there are two variables which describe the surface of the sphere. So geometry of two dimensional space differs from the two dimensional plane. For example the circumference of a circle of radius R on a plane is $C=2\pi R$.

But if same circle of radius R is drawn on a sphere in space, its circumference will not be simply

$C = 2\pi R$. Rather it will be

$$C = \int_0^{2\pi} R\sin\theta d\phi = R\sin\theta|\phi|_0^{2\pi} = 2\pi R\sin\theta$$

Similarly the sum of angles in a triangle on the plane is 180°. The sum of angles in a triangle drawn on the surface of the sphere is not 180° rather it is 270°

Thus the geometry of a spherical surface is different from that of a plane is constructed by three great circles. Great circle is a circle drawn on surface of sphere obtained by the intersection of the sphere and a plane passing through the center of sphere.

Here again geometry of space is different from plane. The fact behind this difference is that plane is flat and surface is curved.

5.9.3 Line Element and Metric Tensor

In n dimensional Riemannian space the line element between two neighbouring points can be defined using metric tensor as

$$d\ell^2 = g_{\mu\nu}dq^\mu dq^\nu$$

Where $g^{\mu\nu}$ are coefficients of the general symmetric tensor and q^μ and q^ν are general coordinates.

For Cartesian coordinates (x, y) in Euclidean plane line element in terms of metric tensor becomes

$$d\ell^2 = g_{11}dq^1dq^1 + g_{12}dq^1dq^2 + g_{21}dq^2dq^1 + g_{22}dq^2dq^2$$

As

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $q^1 = x$ and $q^2 = y$. So

$$\begin{aligned} d\ell^2 &= (1)\,dxdx + (0)\,dxdy + (0)\,dydx + (1)\,dydy \\ d\ell^2 &= dx^2 + dy^2 \\ d\ell &= \sqrt{dx^2 + dy^2} \end{aligned}$$

The line element can also be written as

$$d\ell^2 = \delta_{ij}dq^idq^j$$

Where δ_{ij} is Kronecker delta given by

$$\delta_{ij} = \begin{cases} 1 \text{if} i = j \\ 0 \text{if} i \neq j \end{cases}$$

Thus in two dimensions $g_{ij} = \delta_{ij}$.

For plane polar coordinates (r, θ) the line element will be

$$d\ell^2 = g_{11}dq^1dq^1 + g_{12}dq^1dq^2 + g_{21}dq^2dq^1 + g_{22}dq^2dq^2$$

As

$$g_{ij} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

and $q^1 = r$ and $q^2 = \theta$. So

$$\begin{aligned} d\ell^2 &= (1)\,drdr + (0)\,drd\theta + (0)\,d\theta dr + \left(r^2\right)d\theta d\theta \\ d\ell^2 &= dr^2 + r^2d\theta^2 \\ d\ell &= \sqrt{dr^2 + r^2d\theta^2} \end{aligned}$$

In Euclidean space metric tensor is

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The line element in the form of Cartesian coordinates (x, y, z) will be

$$\begin{aligned} d\ell^2 &= (1)\,dxdx + (0)\,dxdy + (0)\,dxdz + (0)\,dydx + (1)\,dydy \\ &\quad + (0)\,dydz + (0)\,dzdx + (0)\,dzdy + (1)\,dzdz \\ d\ell^2 &= dx^2 + dy^2 + dz^2 \\ d\ell &= \sqrt{dx^2 + dy^2 + dz^2} \end{aligned}$$

In spherical coordinates (r, θ, ϕ) metric tensor is

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

So the line element in the form of Spherical coordinates will be

$$\begin{aligned} d\ell^2 &= (1)\,drdr + (0)\,drd\theta + (0)\,drd\phi + (0)\,d\theta dr + \left(r^2\right)d\theta d\theta \\ &\quad + (0)\,d\theta d\phi + (0)\,d\phi dr + (0)\,d\phi d\theta + \left(r^2\sin^2\theta\right)d\phi d\phi \\ d\ell^2 &= dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \\ d\ell &= \sqrt{dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2} \end{aligned}$$

In cylindrical coordinates (ρ, ϕ, z) metric tensor is

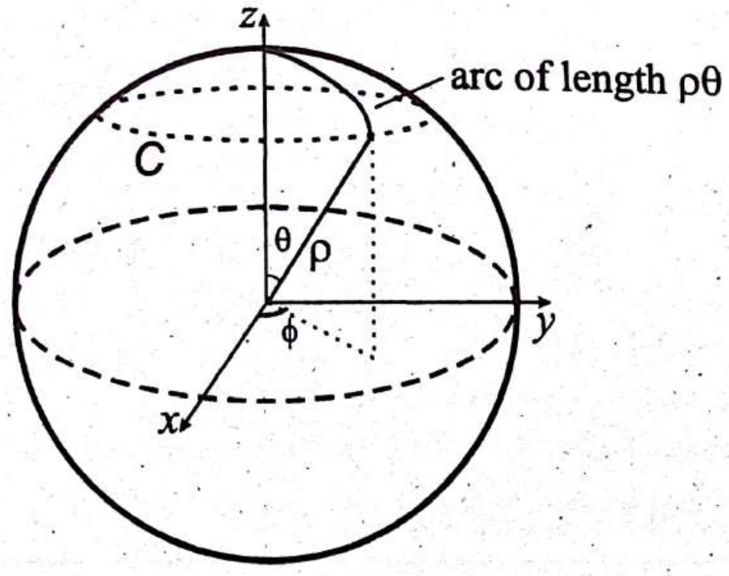


Figure 5.4: *Line element in cylindrical coordinates*

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So the line element in the form of Cylindrical coordinates (ρ, ϕ, z) will be

$$\begin{aligned} d\ell^2 &= (1)\, d\rho d\rho + (0)\, d\rho d\phi + (0)\, d\rho dz + (0)\, d\phi d\rho + \left(\rho^2\right) d\phi d\phi \\ &\quad + (0)\, d\phi dz + (0)\, dz d\rho + (0)\, dz d\phi + (1)\, dz dz \\ d\ell^2 &= d\rho^2 + \rho^2 d\phi^2 + dz^2 \\ d\ell &= \sqrt{d\rho^2 + \rho^2 d\phi^2 + dz^2} \end{aligned}$$

In Minkowski space metric tensor $g_{\mu\nu}$ is replaced by Minkowski matrix $\eta_{\mu\nu}$ which is given by

$$\eta_{\mu\nu} = \begin{pmatrix} \eta_{00} & \eta_{01} & \eta_{02} & \eta_{03} \\ \eta_{10} & \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{20} & \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{30} & \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The sum of diagonal entries is called **signature of the metric**. The signature of above metric is 2.

Thus the line element in metric form is

$$d\ell^2 = \eta_{\mu\nu} dq^\mu dq^\nu$$

As $q^o = ct, q^1 = x, q^2 = y$ and $q^3 = z$. So

$$\begin{aligned} d\ell^2 &= -(cdt)^2 - dx^2 - dy^2 - dz^2 = d\vec{r}^2 - c^2dt^2 \\ d\ell &= \sqrt{\mathrm{dr}^2 - c^2dt^2} \end{aligned}$$

The line element in Minkowski space interms of spherical coordinates (ct, r, θ, ϕ) can be written using the following metric tensor

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

5.10 Geodesics

Q. What are geodesics? Describe its types.

The shortest distance between two points on a curved surface is called geodesic.

In a flat space the shortest distance between two points is a straight line called displacement but in curved space it is more than a straight line which changes its direction at each point of the curve which can be determined by a tangent vector. It is the geodesic which is analogue of straight line in curved space. For instance, a great circle is a geodesic on the surface of a sphere.

5.10.1 Timelike Geodesics

If parameter α is proportional to the arc length of curve then the geodesic is called timelike geodesic.

In a Riemann manifold, a geodesic joining two points attains the minimum length among neighbouring curves joining these points.

In spacetime the paths of the particles with mass moving in the vicinity of spherical massive object are timelike geodesics.

It has negative norm

$$|u| = g_{\mu\nu}u^{\mu}u^{\nu}$$

Particles moving on timelike curves are particles of $p^2 = -M^2 < 0$ generally called massive particles or tardyons which moves more slowly than light.

5.10.2 Spacelike Geodesics

Particles moving on spacelike geodesics are particles of $p^2 = -M^2 > 0$ generally called tachyons.

5.10.3 Null Geodesics

The paths followed by photons in spacetime are null geodesics.

Particles moving on null geodesic are particles of $p^2 = -M^2 = 0$ generally called massless particles or luxons.

5.11 Geodesic Equation

Q. Derive geodesic equation in general relativity.

An equation which describes the geodesic in the curved spacetime is called geodesic equation.

5.11.1 Derivation from Affine Parameter

Let a point P on a curve $x^{\mu}(\alpha)$ parameterized by a parameter α. At this point the direction of the curve can be determined using a tangent vector a (Fig.5.5) given by

$$\mathbf{a} = a^{\mu} e_{\mu}$$

provided the parallel transport of the tangent vector along the curve from α to $\alpha + d\alpha$ gives the vector proportional to tangent vector a. i.e.,

$$\frac{d\mathbf{a}}{d\alpha} = f(\alpha)\,\mathbf{a}$$

In case of geodesic absolute vector will be used rather than absolute vector. i.e.,

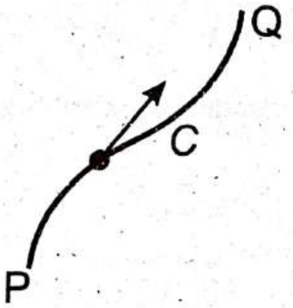


Figure 5.5: *The tangent vector t to the curve C at the point P.*

$$\begin{aligned} \frac{Da^{\mu}}{D\alpha} &= f(\alpha)\,a^{\mu} \\ \frac{da^{\mu}}{d\alpha} + \Gamma^{\mu}_{\nu\lambda} a^{\nu} \frac{dx^{\lambda}}{d\alpha} &= f(\alpha)\,a^{\mu} \end{aligned}$$

As tangent vector is related with position vector as $a^{\mu} = \frac{dx^{\mu}}{d\alpha}$, then

$$\begin{aligned} \frac{d}{d\alpha}\left(\frac{dx^{\mu}}{d\alpha}\right) + \Gamma^{\mu}_{\nu\lambda} \frac{dx^{\nu}}{d\alpha}\frac{dx^{\lambda}}{d\alpha} &= f(\alpha)\frac{dx^{\mu}}{d\alpha} \\ \frac{d^2x^{\mu}}{d\alpha^2} + \Gamma^{\mu}_{\nu\lambda} \frac{dx^{\nu}}{d\alpha}\frac{dx^{\lambda}}{d\alpha} &= f(\alpha)\frac{dx^{\mu}}{d\alpha} \end{aligned}$$

Choosing α as an affine parameter for which $f(\alpha) = 0$ above equation becomes

$$\frac{d^2x^\mu}{d\alpha^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha} = 0$$

This is called **geodesic equation.** It plays the same in General relativity as equation of motion in Newtonian mechanics.

For timelike paths, interms of four velocity $u^\mu = \dfrac{dx^\mu}{d\tau}$ geodesic equation becomes

$$u^\nu \nabla_\nu u^\mu = 0$$

Similarly interms of four momentum $p^\mu = mu^\mu$ geodesic equation becomes

$$p^\nu \nabla_\nu p^\mu = 0$$

5.11.2 Derivation from Action

In two dimensional Euclidean plane the length of the geodesic is given by

$$\begin{aligned} \ell &= \int d\ell = \int \sqrt{dx^2 + dy^2} \\ \ell &= \int d\ell = \int \sqrt{\left(\frac{dx}{d\alpha}\right)^2 + \left(\frac{dy}{d\alpha}\right)^2} d\alpha \end{aligned}$$

In Riemannian geometry the length of the geodesic is given by

$$\begin{aligned} \ell &= \int \sqrt{g_{\mu\nu} dx^\mu dx^\nu} \\ \ell &= \int \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\alpha}\frac{dx^\nu}{d\alpha}} d\alpha \equiv \int \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} d\alpha \\ \ell &= \int \mathcal{L}(x^\mu, \dot{x}^\mu)\, d\alpha \end{aligned}$$

Where $\mathcal{L}(x^\mu, \dot{x}^\mu) = \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$ is a function of position and velocity and defines the path between two points on a curve.

To find the shortest path (geodesic) between two points of a curve, minimize ℓ. i.e., $\delta\ell = 0$. So

$$\delta \int_{\alpha_1}^{\alpha_2} \mathcal{L}(x^\mu, \dot{x}^\mu)\, d\alpha = \int_{\alpha_1}^{\alpha_2} \delta\mathcal{L}(x^\mu, \dot{x}^\mu)\, d\alpha = 0$$

Using Leibnitz theorem of partial differentiation

$$\delta\mathcal{L}(x^\mu, \dot{x}^\mu) = \frac{\partial \mathcal{L}}{\partial x^\mu}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\delta \dot{x}^\mu$$

Above integral becomes

$$
\begin{aligned}
\int_{\alpha_1}^{\alpha_2}\left(\frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu+\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\delta\dot{x}^\mu\right)d\alpha &= 0\\
\int_{\alpha_1}^{\alpha_2}\left(\frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu+\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\delta\frac{dx^\mu}{d\alpha}\right)d\alpha &= 0\\
\int_{\alpha_1}^{\alpha_2}\frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu d\alpha+\int_{\alpha_1}^{\alpha_2}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\delta\frac{dx^\mu}{d\alpha}d\alpha &= 0
\end{aligned}
$$

Integrating second term by parts

$$
\int_{\alpha_1}^{\alpha_2}\frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu d\alpha+\left[\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}|\delta x^\mu|_{\alpha_1}^{\alpha_2}-\int_{\alpha_1}^{\alpha_2}\frac{d}{d\alpha}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right)\delta x^\mu d\alpha\right]=0
$$

At boundary values the derivative vanishes. So

$$
\begin{aligned}
\int_{\alpha_1}^{\alpha_2}\frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu d\alpha+\left[\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}(0)-\int_{\alpha_1}^{\alpha_2}\frac{d}{d\alpha}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right)\delta x^\mu d\alpha\right] &= 0\\
\int_{\alpha_1}^{\alpha_2}\frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu d\alpha-\int_{\alpha_1}^{\alpha_2}\frac{d}{d\alpha}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right)\delta x^\mu d\alpha &= 0\\
\int_{\alpha_1}^{\alpha_2}\left[\frac{\partial\mathcal{L}}{\partial x^\mu}-\frac{d}{d\alpha}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right)\right]\delta x^\mu d\alpha &= 0\\
\frac{\partial\mathcal{L}}{\partial x^\mu}-\frac{d}{d\alpha}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right) &= 0
\end{aligned}
$$

This is Euler-Lagrange equation. This equation can be led to geodesic equation by making a suitable substitution of the Lagrangian function $(\mathcal{L})$

$$\mathcal{L}=g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

Differentiating with respect to x^μ and x^ν

$$\frac{\partial\mathcal{L}}{\partial x^\mu}=\frac{\partial g_{\lambda\rho}}{\partial x^\mu}\dot{x}^\lambda\dot{x}^\rho,\qquad \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}=2g_{\mu\nu}\dot{x}^\nu$$

Putting in Euler-Lagrange equation

$$\frac{\partial g_{\lambda\rho}}{\partial x^\mu}\dot{x}^\lambda\dot{x}^\rho-\frac{d}{d\alpha}\left(2g_{\mu\nu}\dot{x}^\nu\right) = 0$$

$$\begin{aligned}
\frac{1}{2}\frac{\partial g_{\lambda\rho}}{\partial x^\mu}\dot{x}^\lambda\dot{x}^\rho - \left(g_{\mu\nu}\frac{d\dot{x}^\nu}{d\alpha} + \frac{\partial g_{\mu\nu}}{\partial\alpha}\dot{x}^\nu\right) &= 0 \\
\frac{1}{2}\frac{\partial g_{\lambda\rho}}{\partial x^\mu}\frac{dx^\lambda}{d\alpha}\frac{dx^\rho}{d\alpha} - g_{\mu\nu}\frac{d^2x^\nu}{d\alpha^2} - \frac{\partial g_{\mu\nu}}{\partial\alpha}\frac{dx^\nu}{d\alpha} &= 0 \\
\frac{1}{2}\frac{\partial g_{\lambda\rho}}{\partial x^\mu}\frac{dx^\lambda}{d\alpha}\frac{dx^\rho}{d\alpha} - g_{\mu\nu}\frac{d^2x^\nu}{d\alpha^2} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{dx^\lambda}{d\alpha}\frac{dx^\nu}{d\alpha} &= 0
\end{aligned}$$

Changing dummy indices to make terms similar

$$\begin{aligned}
\frac{1}{2}\frac{\partial g_{\lambda\rho}}{\partial x^\mu}\frac{dx^\lambda}{d\alpha}\frac{dx^\rho}{d\alpha} - g_{\mu\rho}\frac{d^2x^\rho}{d\alpha^2} - \frac{\partial g_{\mu\rho}}{\partial x^\lambda}\frac{dx^\lambda}{d\alpha}\frac{dx^\rho}{d\alpha} &= 0 \\
\frac{1}{2}\frac{\partial g_{\lambda\rho}}{\partial x^\mu}\frac{dx^\lambda}{d\alpha}\frac{dx^\rho}{d\alpha} - g_{\mu\rho}\frac{d^2x^\rho}{d\alpha^2} - \left(\frac{1}{2}\frac{\partial g_{\mu\rho}}{\partial x^\lambda}\frac{dx^\lambda}{d\alpha}\frac{dx^\rho}{d\alpha} + \frac{1}{2}\frac{\partial g_{\mu\lambda}}{\partial x^\rho}\frac{dx^\lambda}{d\alpha}\frac{dx^\rho}{d\alpha}\right) &= 0 \\
\frac{1}{2}\left(\frac{\partial g_{\mu\rho}}{\partial x^\lambda} + \frac{\partial g_{\mu\lambda}}{\partial x^\rho} - \frac{\partial g_{\lambda\rho}}{\partial x^\mu}\right)\frac{dx^\lambda}{d\alpha}\frac{dx^\rho}{d\alpha} - g_{\mu\rho}\frac{d^2x^\rho}{d\alpha^2} &= 0
\end{aligned}$$

Multiplying by inverse metric tensor $g^{\mu\sigma}$

$$\begin{aligned}
\frac{1}{2}g^{\mu\sigma}\left(\frac{\partial g_{\mu\rho}}{\partial x^\lambda} + \frac{\partial g_{\mu\lambda}}{\partial x^\rho} - \frac{\partial g_{\lambda\rho}}{\partial x^\mu}\right)\frac{dx^\lambda}{d\alpha}\frac{dx^\rho}{d\alpha} - g^{\mu\sigma}g_{\mu\rho}\frac{d^2x^\rho}{d\alpha^2} &= 0 \\
\Gamma^\mu_{\lambda\rho}\frac{dx^\lambda}{d\alpha}\frac{dx^\rho}{d\alpha} - \frac{d^2x^\sigma}{d\alpha^2} &= 0
\end{aligned}$$

Where $\Gamma^\mu_{\lambda\rho} = \frac{1}{2}g^{\mu\sigma}\left(\frac{\partial g_{\mu\rho}}{\partial x^\lambda} + \frac{\partial g_{\mu\lambda}}{\partial x^\rho} - \frac{\partial g_{\lambda\rho}}{\partial x^\mu}\right)$

This is called **geodesic equation.**

5.12 Geodesic Equation of Deviation

Q. Derive geodesic equation of deviation.

The rate of acceleration of distance between nearby geodesic is linked with the curvature by the equation is called the equation of geodesic deviation.
An equation which relates the Riemann tensor to the relative acceleration of two neighbouring geodesics is called equation of geodesic deviation.

In Riemannian geometry it is also called **Jacobi equation.**

Let us consider two neighbouring curves on Riemannian manifold $x^\mu(\alpha)$ and $x^\mu(\alpha) + \eta^\mu(\alpha)$, where α is a parameter. Then tangent vectors to the curves will be

$$a^\mu = \frac{dx^\mu}{d\alpha}$$

Let a connecting vector η which relates two points of the neighbouring curves. Then geodesic equations for both curves will be

$$\frac{d^2x^\mu}{d\alpha^2} + \Gamma^\mu_{\nu\lambda}(x)\frac{dx^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha} = 0 \tag{5.11}$$

and

$$\begin{aligned}
\frac{d^2}{d\alpha^2}(x^\mu + \eta^\mu) + \Gamma^\mu_{\nu\lambda}(x^\mu + \eta^\mu)\frac{d}{d\alpha}(x^\nu + \eta^\nu)\frac{d}{d\alpha}(x^\lambda + \eta^\lambda) &= 0 \\
\frac{d^2x^\mu}{d\alpha^2} + \frac{d^2\eta^\mu}{d\alpha^2} + \Gamma^\mu_{\nu\lambda}(x+\eta)\left(\frac{dx^\nu}{d\alpha} + \frac{d\eta^\nu}{d\alpha}\right)\left(\frac{dx^\lambda}{d\alpha} + \frac{d\eta^\lambda}{d\alpha}\right) &= 0
\end{aligned} \tag{5.12}$$

For small separation η^μ the Christofell symbols $\Gamma^\mu_{\nu\lambda}(x+\eta)$ can be expanded using Taylor's expansion

$$\Gamma^\mu_{\nu\lambda}(x+\eta) = \Gamma^\mu_{\nu\lambda}(x) + \partial_\rho\Gamma^\mu_{\nu\lambda}(x)\,\eta^\rho + \ldots$$

Keeping only first order terms

$$\Gamma^\mu_{\nu\lambda}(x+\eta) \approx \Gamma^\mu_{\nu\lambda}(x) + \partial_\rho\Gamma^\mu_{\nu\lambda}(x)\,\eta^\rho$$

Putting in Eq.(5.12)

$$\begin{aligned}
\frac{d^2x^\mu}{d\alpha^2} + \frac{d^2\eta^\mu}{d\alpha^2} + (\Gamma^\mu_{\nu\lambda} + \partial_\rho\Gamma^\mu_{\nu\lambda}\eta^\rho)\left(\frac{dx^\nu}{d\alpha} + \frac{d\eta^\nu}{d\alpha}\right)\left(\frac{dx^\lambda}{d\alpha} + \frac{d\eta^\lambda}{d\alpha}\right) &= 0 \\
\frac{d^2x^\mu}{d\alpha^2} + \frac{d^2\eta^\mu}{d\alpha^2} + (\Gamma^\mu_{\nu\lambda} + \partial_\rho\Gamma^\mu_{\nu\lambda}\eta^\rho)\left(\frac{dx^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha} + \frac{dx^\nu}{d\alpha}\frac{d\eta^\lambda}{d\alpha} + \frac{d\eta^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha} + \frac{d\eta^\nu}{d\alpha}\frac{d\eta^\lambda}{d\alpha}\right) &= 0 \\
\frac{d^2x^\mu}{d\alpha^2} + \frac{d^2\eta^\mu}{d\alpha^2} + \left(\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\alpha}\frac{d\eta^\lambda}{d\alpha} + \Gamma^\mu_{\nu\lambda}\frac{d\eta^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha} + \Gamma^\mu_{\nu\lambda}\frac{d\eta^\nu}{d\alpha}\frac{d\eta^\lambda}{d\alpha}\right)& \\
+\left(\partial_\rho\Gamma^\mu_{\nu\lambda}\eta^\rho\frac{dx^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha} + \partial_\rho\Gamma^\mu_{\nu\lambda}\eta^\rho\frac{dx^\nu}{d\alpha}\frac{d\eta^\lambda}{d\alpha} + \partial_\rho\Gamma^\mu_{\nu\lambda}\eta^\rho\frac{d\eta^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha} + \partial_\rho\Gamma^\mu_{\nu\lambda}\eta^\rho\frac{d\eta^\nu}{d\alpha}\frac{d\eta^\lambda}{d\alpha}\right) &= 0
\end{aligned}$$

Keeping only terms of the first order of η

$$\frac{d^2x^\mu}{d\alpha^2} + \frac{d^2\eta^\mu}{d\alpha^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\alpha}\frac{d\eta^\lambda}{d\alpha} + \Gamma^\mu_{\nu\lambda}\frac{d\eta^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha} + \partial_\rho\Gamma^\mu_{\nu\lambda}\eta^\rho\frac{dx^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha} = 0 \tag{5.13}$$

Subtracting Eq.(5.11) from Eq.(5.13)

$$\begin{aligned}
\frac{d^2\eta^\mu}{d\alpha^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\alpha}\frac{d\eta^\lambda}{d\alpha} + \Gamma^\mu_{\nu\lambda}\frac{d\eta^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha} + \partial_\rho\Gamma^\mu_{\nu\lambda}\eta^\rho\frac{dx^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha} &= 0 \\
\frac{d^2\eta^\mu}{d\alpha^2} + 2\Gamma^\mu_{\nu\lambda}\frac{d\eta^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha} + \partial_\rho\Gamma^\mu_{\nu\lambda}\eta^\rho\frac{dx^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha} &= 0
\end{aligned} \tag{5.14}$$

Now the covariant derivative of η^μ will be

$$\frac{D\eta^\mu}{d\alpha} = \frac{d\eta^\mu}{d\alpha} + \Gamma^\mu_{\nu\lambda}\eta^\nu\frac{dx^\lambda}{d\alpha} \tag{5.15}$$

Taking covariant derivative with respect to α

$$\begin{aligned}
\frac{D}{d\alpha}\left(\frac{D\eta^\mu}{d\alpha}\right) &= \frac{D}{d\alpha}\left(\frac{d\eta^\mu}{d\alpha}+\Gamma^\mu_{\nu\lambda}\eta^\nu\frac{dx^\lambda}{d\alpha}\right)\\
\frac{D^2\eta^\mu}{d\alpha^2} &= \frac{d}{d\alpha}\left(\frac{D\eta^\mu}{d\alpha}\right)+\Gamma^\mu_{\nu\lambda}\left(\frac{D\eta^\mu}{d\alpha}\right)\frac{dx^\lambda}{d\alpha}\\
&= \frac{d}{d\alpha}\left(\frac{d\eta^\mu}{d\alpha}+\Gamma^\mu_{\nu\lambda}\eta^\nu\frac{dx^\lambda}{d\alpha}\right)+\Gamma^\mu_{\nu\lambda}\left(\frac{d\eta^\mu}{d\alpha}+\Gamma^\mu_{\nu\lambda}\eta^\nu\frac{dx^\lambda}{d\alpha}\right)\frac{dx^\lambda}{d\alpha}
\end{aligned}$$

Again using covariant derivative from Eq.(5.15)

$$\frac{D^2\eta^\mu}{d\alpha^2}=\frac{d^2\eta^\mu}{d\alpha^2}+\partial_\rho\Gamma^\mu_{\nu\lambda}\frac{dx^\rho}{d\alpha}\eta^\nu\frac{dx^\lambda}{d\alpha}+\Gamma^\mu_{\nu\lambda}\frac{d\eta^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha}+\Gamma^\mu_{\nu\lambda}\eta^\nu\frac{d^2x^\lambda}{d\alpha^2}+\Gamma^\mu_{\nu\lambda}\frac{d\eta^\mu}{d\alpha}\frac{dx^\lambda}{d\alpha}+\Gamma^\mu_{\nu\lambda}\Gamma^\mu_{\rho\sigma}\eta^\rho\frac{dx^\sigma}{d\alpha}\frac{dx^\lambda}{d\alpha}$$

Using Eq.(5.11) for fourth term and Eq.(5.14) for first term

$$\begin{aligned}
\frac{D^2\eta^\mu}{d\alpha^2} &= -2\Gamma^\mu_{\nu\lambda}\frac{d\eta^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha}-\partial_\rho\Gamma^\mu_{\nu\lambda}\eta^\rho\frac{dx^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha}+\partial_\rho\Gamma^\mu_{\nu\lambda}\frac{dx^\rho}{d\alpha}\eta^\nu\frac{dx^\lambda}{d\alpha}+\Gamma^\mu_{\nu\lambda}\frac{d\eta^\mu}{d\alpha}\frac{dx^\lambda}{d\alpha}\\
&\quad+\Gamma^\mu_{\nu\lambda}\eta^\nu\left(-\Gamma^\lambda_{\rho\sigma}\frac{dx^\rho}{d\alpha}\frac{dx^\sigma}{d\alpha}\right)+\Gamma^\mu_{\nu\lambda}\frac{d\eta^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha}+\Gamma^\mu_{\nu\lambda}\Gamma^\mu_{\rho\sigma}\eta^\rho\frac{dx^\sigma}{d\alpha}\frac{dx^\lambda}{d\alpha}\\
&= -2\Gamma^\mu_{\nu\lambda}\frac{d\eta^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha}-\partial_\rho\Gamma^\mu_{\nu\lambda}\eta^\rho\frac{dx^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha}+\partial_\rho\Gamma^\mu_{\nu\lambda}\frac{dx^\rho}{d\alpha}\eta^\nu\frac{dx^\lambda}{d\alpha}\\
&\quad+2\Gamma^\mu_{\nu\lambda}\frac{d\eta^\mu}{d\alpha}\frac{dx^\lambda}{d\alpha}-\Gamma^\mu_{\nu\lambda}\eta^\nu\Gamma^\lambda_{\rho\sigma}\frac{dx^\rho}{d\alpha}\frac{dx^\sigma}{d\alpha}+\Gamma^\mu_{\nu\lambda}\Gamma^\mu_{\rho\sigma}\eta^\rho\frac{dx^\sigma}{d\alpha}\frac{dx^\lambda}{d\alpha}\\
&= -\partial_\rho\Gamma^\mu_{\nu\lambda}\eta^\rho\frac{dx^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha}+\partial_\rho\Gamma^\mu_{\nu\lambda}\frac{dx^\rho}{d\alpha}\eta^\nu\frac{dx^\lambda}{d\alpha}-\Gamma^\mu_{\nu\lambda}\eta^\nu\Gamma^\lambda_{\rho\sigma}\frac{dx^\rho}{d\alpha}\frac{dx^\sigma}{d\alpha}+\Gamma^\mu_{\nu\lambda}\Gamma^\mu_{\rho\sigma}\eta^\rho\frac{dx^\sigma}{d\alpha}\frac{dx^\lambda}{d\alpha}
\end{aligned}$$

Changing dummy indices

$$\begin{aligned}
\frac{D^2\eta^\mu}{d\alpha^2} &= -\partial_\rho\Gamma^\mu_{\nu\lambda}\eta^\rho\frac{dx^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha}+\partial_\lambda\Gamma^\mu_{\rho\nu}\frac{dx^\nu}{d\alpha}\eta^\rho\frac{dx^\lambda}{d\alpha}-\Gamma^\mu_{\rho\sigma}\eta^\rho\Gamma^\sigma_{\nu\lambda}\frac{dx^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha}+\Gamma^\mu_{\sigma\lambda}\Gamma^\sigma_{\rho\nu}\eta^\rho\frac{dx^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha}\\
&= \left(-\partial_\rho\Gamma^\mu_{\nu\lambda}+\partial_\lambda\Gamma^\mu_{\rho\nu}-\Gamma^\mu_{\rho\sigma}\Gamma^\sigma_{\nu\lambda}+\Gamma^\mu_{\sigma\lambda}\Gamma^\sigma_{\rho\nu}\right)\eta^\rho\frac{dx^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha}\\
\frac{D^2\eta^\mu}{d\alpha^2} &= -R^\mu_{\nu\lambda\rho}\eta^\rho\frac{dx^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha}
\end{aligned}$$

Where $R^\mu_{\nu\lambda\rho}=-\partial_\rho\Gamma^\mu_{\nu\lambda}+\partial_\lambda\Gamma^\mu_{\rho\nu}-\Gamma^\mu_{\rho\sigma}\Gamma^\sigma_{\nu\lambda}+\Gamma^\mu_{\sigma\lambda}\Gamma^\sigma_{\rho\nu}$ is Riemann tensor.

This equation is called **equation of geodesic deviation.**

KEY TERMS

- **Tensor:** A mathematical object analogous to vector represented by an array of components that are functions of the space coordinates is called tensor.
- **Rank of a tensor:** The number which shows the dimensions of tensor in array form is called rank of tensor.
- **One form:** A covariant tensor of rank 1 is called one-form.
- **Tensor field:** A function of points in Euclidean space or manifold that defines the value of a field using tensors is called tensor field.
- **Parallel transport:** The method of transporting geometrical data along a curve in manifold preserving the direction of vectors of the manifold with respect to connection is called parallel transport.
- **Metric tensor:** A covariant symmetric tensor of rank 2 that defines the mapping of scalar product between two vectors is called covariant metric tensor.
- **Covariant Differentiation:** A way of specifying a derivative along tangent vector of a manifold is called covariant differentiation.
- **Christofell symbols:** An array of numbers describing a metric connection is called christofell symbol.
- **Riemann curvature tensor:** A tensor that defines the tensor filed at each point in Riemannian manifold is called Riemannian tensor.
- **Ricci tensor:** A tensor which is used to define the part of the curvature of spacetime in Einstein's tensor is called Ricci tensor.
- **Einstein's tensor:** A tensor used to express the curvature of pseudo Riemannian manifold in differential geometry is called Einstein's tensor.
- **Riemannian geometry:** The branch of mathematics which deals with the study of Riemannian manifolds and smooth manifolds with a Riemannian metric is called Riemannian geometry.
- **Manifold:** A continuous space of points that resembles with Euclidean space near each point and maybe curved globally and flat locally is called manifold.
- **Line element:** A line segment associated with an infinitesimal displacement vector in a metric space is called line element.
- **Geodesic:** The shortest distance between two points on a curved surface is called geodesic.
- **Timelike Geodesic:** If parameter α is proportional to the arc length of curve then the geodesic is called timelike geodesic

- **Spacelike Geodesic:** Particles moving on spacelike geodesics are particles of $p^2 = -M^2 > 0$ generally called tachyons.
- **Null geodesic:** The paths followed by photons in spacetime are null geodesics
- **Geodesic equation:** An equation which describes the geodesic in the curved spacetime is called geodesic equation.

KEY FORMULAE

Christofell symbols

$$\frac{\partial e_\mu}{\partial x^\nu} = \sum_\lambda \Gamma^\lambda_{\mu\nu} e_\lambda$$

Riemann Curvature Tensor

$$R^\rho_{\mu\nu\lambda} = \partial_\nu \Gamma^\rho_{\mu\lambda} - \partial_\lambda \Gamma^\rho_{\mu\nu} + \Gamma^\sigma_{\mu\lambda}\Gamma^\rho_{\sigma\nu} - \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\lambda}$$

Bianchi identity

$$\nabla_\mu R_{\rho\sigma\nu\lambda} + \nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\nu R_{\rho\sigma\lambda\mu} = 0$$

Ricci tensor

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} g^{\lambda\rho} R_{\mu\nu\lambda\rho} = R^\mu_\mu$$

Einstein's Tensor

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$$

Line element (Cartesian)

$$d\ell = \sqrt{dx^2 + dy^2}$$

Line element (plane polar)

$$d\ell = \sqrt{r^2 d\theta^2 + dr^2}$$

Line element (Spherical)

$$\ell = \int \sqrt{dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2}$$

Line element (Cylindrical)

$$d\ell = \sqrt{d\rho^2 + \rho^2 d\phi^2 + dz^2}$$

Line element in metric form

$$d\ell^2 = g_{\mu\nu} dq^\mu dq^\nu$$

Geodesic equation

$$\frac{d^2 x^\mu}{d\alpha^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha} = 0$$

Geodesic Equation of Deviation

$$\frac{D^2 \eta^\mu}{d\alpha^2} = -R^\mu_{\nu\lambda\rho} \eta^\rho \frac{dx^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha}$$

EXERCISE

Problems

1. Show that covariant derivative of a vector field i.e. $\nabla_\nu X^\mu$ represents a $(1,1)$ tensor if the connection component do not transform as a $(1,2)$ tensor

2. Find the geodesic equations for R^3 in cylindrical polar coordinates where the line element is $ds^2 = dR^2 + R^2 d\phi^2 + dz^2$

3. Consider the line element $ds^2 = a^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$, where a is the radius of the sphere. Calculate $\Gamma^\theta_{\phi\phi}, \Gamma^\phi_{\theta\phi}$ and $\Gamma^\phi_{\phi\theta}$

4. Show that the components of four acceleration are given by

$$Q^\mu = \left(\frac{\vec{v}.\vec{Q}}{c}, \gamma^2\vec{a} + \gamma v\left(\frac{\vec{v}\cdot\vec{a}}{c^2}\right)\right)$$

5. In a nuclear fission of uranium-235 caused by the absorption of a neutron, nuclei of krypton and barium are produced, and three neutrons are emitted. The difference in the total mass of the particles present at the start of the process and those present at the end is $3.08 \times 10^{-28} kg$. What is the energy released in this process, in both joules and electronvolts ($1eV = 1.60 \times 10^{-19} J$)?

6. Given that the binding energy of a hydrogen atom is 13.6 eV, what is the difference between the mass of a hydrogen atom and the masses of its constituent electron and proton?

7. In frame of reference S, an electron moving along the x-axis has energy $3m_e c^2$ and momentum magnitude $\sqrt{8}m_e c$. Use the transformations of energy and momentum to find the energy and momentum magnitude observed in frame S' moving with speed $4c/5$ relative to S in the positive x-direction.

8. If $T^{\mu\nu}$ and S_μ are $(2,0)$ and $(0,1)$ tensors respectively, then find the rank of tensor $T^{\mu\nu}S_\nu$.

9. Show that $\partial_\mu V^\nu$ is not a $(1,1)$ tensor.

10. Find the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ corresponding to the line element

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

11. Transform $ds^2 = dx^2 + dy^2 + dy^2$ into spherical polar coordinates

12. Let the metric tensor is given by $ds^2 = dp^2 - pdpdq + q^2dq^2$ with $x^o = p$ and $x^1 = q$, then calculate the Lie derivative of $T^a = (-q, 2p)$ along $S_a = (1, -1)$

13. Show that $\nabla_{[c}\nabla_{d]}X^a = \frac{1}{2}R^a_{bcd}X^b$, where R^a_{bcd} is Riemann Tensor

14. The Riemann curvature tensor has components

$$R^{\mu}_{\nu\lambda\sigma} = 2\Gamma^{\mu}_{\nu[\sigma,\lambda]} - 2\Gamma^{\mu}_{\rho[\sigma}\Gamma^{\rho}_{|\nu|\lambda]}$$

If $\Gamma^{\mu}_{\nu\lambda} = 0$ at point P, then show that $R^{\mu}_{\nu\sigma\rho} = \frac{2}{3}\left[R^{\mu}_{(\nu\sigma)\rho} - R^{\mu}_{(\nu\rho)\sigma}\right]$

15. A contravariant tensor is given by $\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$. Transform this tensor into Cylindrical polar coordinates.

16. The metric tensor for $x^1 = q, x^2 = p$ is $\begin{bmatrix} q^2 & 0 \\ 1 & p^2 \end{bmatrix}$. Determine the Lie and Covariant derivatives of $T^a = (q, -p)$ along $S_h = (0, -q)$

17. Let the metric tensor is given by $ds^2 = dp^2 + q^2 dq^2$ with $x^0 = p$ and $x^1 = q$, the calculate the Lie derivative of $T^a = (-q, 2p)$ along $S_a = (1, -1)$

18. If $X^{\mu\nu}$ is anti-symmetric and $Y_{\mu\nu}$ is symmetric, then show that $X^{\mu\nu}Y_{\mu\nu} = 0$

19. Find the Christoffel symbols $\Gamma^{r}_{\phi\phi}$ and Γ^{r}_{00}, also find the equations of geodesic corresponding to $\Gamma^{r}_{\phi\phi}$ and Γ^{r}_{00} for the metric

$$ds^2 = dt^2 - a^2(t)\left[\frac{1}{1-kr^2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right]$$

Chapter 6

General Theory of Relativity

Learning Objectives

In this chapter you will learn

- The difference between special and general theory of relativity.
- About curved spacetime
- Newton's law of gravitation
- Inertial and gravitational mass
- The equivalence principle and its forms
- Stress-Energy tensor
- Einstein's field equations
- Riemann curvature tensor
- Ricci tensor
- Manifolds
- Line element and its types
- Geodesic
- Geodesic equation

KEY TERMS

curved spacetime, inertial mass, gravitational mass, stress-energy tensor, energy density, momentum density, energy flux, stress, dust, ideal fluid,

6.1 Difference between Special and General Relativity

Q. Write differences between special and general relativity.
Special theory of relativity was announced by Einstein in 1905 but it lacks the concept about gravitation. In 1916, he announced the General theory of relativity containing the concept of gravitation. There are some differences between these theories which have been summarized below:

Special Relativity	General Relativity
It deals with the velocity difference of two objects in inertial frames.	It deals with the acceleration difference between two objects in non-inertial frames.
Special relativity explains that spacetime is flat.	General relativity explains that spacetime is curved. The curvature of the spacetime depends on mass.
Things that are moving at the speed of light will always move at the speed of light in comparison to you, regardless how fast you are showing your movement	Gravity is capable of bending light, radio waves, and several other things.
It is simple, not detailed and did not cover the whole universe.	It is complex, comprehensive and covered greater part of the universe

6.2 Curved Spacetime

Q. Write a note on curved spacetime
In 1907, Minkowski introduced the concept of spacetime but that was actually flat spacetime. When Einstein considered the geometrical explanation using flat spacetime, he concluded that spacetime is only flat when there is no mass. As an object of mass m is placed spacetime becomes curve. The curvature of spacetime depends on the mass of the object. In special relativity the curvature of Minkowski flat spacetime is zero.

The coordinates of curved spacetime are position dependent while that of flat spacetime are position independent. Curved spacetime is described by Riemannian geometry.According to Einstein gravity is nothing more than a curved space.

6.3 Newton's Gravitational Law

Q. What is Newton's gravitational law?

The gravitational force between two objects is directly proportional to the product of their masses and inversely proportional to the square of the separation between these objects.

Mathematically for two objects of masses m_1 and m_2 separated by r, the gravitational force will be

$$F \propto m_1 m_2$$

$$F \propto \frac{1}{r^2}$$

Combining above two relations

$$\begin{aligned} F &\propto \frac{m_1 m_2}{r^2} \\ F &= G\frac{m_1 m_2}{r^2} \end{aligned}$$

Where G is gravitational constant and its value is $6.67 \times 10^{-11} Nm^2kg^{-2}$.

6.3.1 Problems with Newton's Gravitational Theory

- In Newton's gravitational law force between two objects is quantitatively described but it is not explained how the force will be mediated.
- Newton's law does not explain the precession of the perihelion of orbits of planets.
- Calculations of bending of light is incorrect.

To compensate above discrepancies, Einstein developed general theory of relativity.

6.4 Mass

Q. What is mass? Also describe its types.

Mass is both a property of a physical body and a measure of its resistance to acceleration when a net force is applied.

6.4.1 Inertial Mass

The mass that determines the acceleration of a body under the action of a given force is called inertial mass.

It is given by Newton's second law of motion

$$m = \frac{\mathbf{F}}{\mathbf{a}}$$

Where **F** is the net force.

Inertial mass is actually inertial resistance to acceleration of the body in the response to all the types of forces.

For example if an object is being pulled by gravity keeping net force constant and when inertial mass is small then acceleration produced will be large and when inertial mass is increased then acceleration produced will becomes small. So inertial mass is not based on how much an object is pulled by the other object but it is based on the inertia that the object has.

Inertial mass is responsible for the interactions of objects and systems. When a particle or particles in a system interact with each other they produce a net force and in the result an acceleration is produced. With zero net force there will be no acceleration. Once we have know the force then we know the acceleration according to Newton's second law of motion.

6.4.2 Gravitational Mass

The mass that determines the acceleration of a body under the action of the gravitational force is called gravitational mass.

It is given by Newton's gravitational law

$$m_g = \frac{\mathbf{F}_g}{\mathbf{g}}$$

Where $\mathbf{F}_g$ is the gravitational force.

Inertial mass and gravitational mass are related by

$$\mathbf{a} = \left(\frac{m_g}{m}\right)\mathbf{g}$$

6.5 The Equivalence Principle

Q. What is the equivalence principle? Also describe its two forms.
A principle based on experiments performed by Galileo and Einstein's observation of gravitational forces formulated the equivalence principle.

6.5.1 Weak Form of Equivalence Principle

All bodies experience the same acceleration when falling freely in the same gravitational field.

Its corollary is

Gravitational mass and inertial mass are equal.

Experiments of Galileo and Newton has shown that inertial and gravitational mass are equivalent. The exact equality of inertial and gravitational mass is called **the weak principle of equivalence**. This principle only discuss the gravitational interaction. But a general principle is needed to include all the types of interactions. Freely falling particles move on timelike geodesics in the spacetime.

6.5.2 Strong Form of Equivalence Principle

The law of physics are the same in an accelerated frame and in a uniform and static gravitational field.

The weak equivalence principle was extended to all the types of interactions by Einstein.

A mental experiment which can be performed if technology is available is called thought experiment.

6.6 Stress-Energy Tensor

Q. What is stress-energy tensor? Describe its different possible components in details.

A second rank tensor that describes the density and flux of energy and momentum in spacetime is called stress-energy tensor.

It is denoted by $T^{\mu\nu}$. It is also called energy-momentum tensor. The stress-energy momentum tensor acts as a source of gravitational field. It describe the flow of energy and momentum in a region of spacetime. It will help to construct Einstein field equation in general relativity. It is related to Einstein tensor $(G^{\mu\nu})$ as

$$G^{\mu\nu} = 8\pi G T^{\mu\nu}$$

Stress-energy tensor is a symmetric tensor. i.e.,

$$T^{\mu\nu} = T^{\nu\mu}$$

It has total sixteen components but due to symmetry six components reduces and ten components left behind that contains four diagonal components and six off-diagonal components.

Covariant form of stress-energy tensor can be written as

$$T_{\mu\nu} = g_{\mu\alpha} g_{\beta\nu} T^{\alpha\beta}$$

Stress-energy tensor is primarily given by

$$T^{\mu\nu} = c\frac{dp^{\mu}}{dS^{\nu}} \tag{6.1}$$

Where p^{μ} is four momentum vector given by

$$p^{\mu} = (p^0, p^i) = \left(\frac{E}{c}, \vec{p}\right)$$

and ΔS^{ν} is Minkowski volume given by $S^{\nu} = (S^0, S^i) = (xyz, ctyz, ctxz, ctxy)$.

Consider a surface defined by constant x^{ν} then $T^{\mu\nu}$ is the flux of the μ components of momentum crossing the interface defined by constant x^μ. Different components of $T^{\mu\nu}$ are T^{00}, T^{01}, T^{i0} and T^{ij} which are described below.

6.6.1 Energy Density

Energy per unit volume is called energy density.

The component T^{00} represents the mass-energy density. To find T^{00} put $\mu = \nu = 0$ in Eq.(6.1)

$$T^{00} = c\frac{dp^0}{dS^0} = \frac{dE}{dxdydz} = \frac{c^2 dm}{dV} = \rho c^2$$

The energy in a volume V can be determined using T^{00} component as

$$\begin{aligned} T^{00} &= \frac{dE}{dV} \\ dE &= T^{00} dV \\ E &= \int_V T^{00}(t, x)\, dV \end{aligned}$$

In fact it is the zero component of four momentum vector. So its more general form is

$$p^\mu = \int_V T^{\mu\nu} \varepsilon_{\nu\lambda\rho\sigma} dx^\lambda dx^\rho dx^\sigma$$

6.6.2 Momentum Density

Momentum per unit volume is called momentum density.

The component T^{i0} represents the momentum density. To find T^{i0} put $\mu = i, \nu = 0$ in Eq.(6.1)

$$T^{i0} = c\frac{dp^i}{dS^0} = c\left(\frac{d\vec{p}}{dxdydz}\right) = c\left(\frac{d\vec{p}}{dV}\right) \equiv \pi^i$$

where π^i is the momentum density. It can also be written in terms of time coordinate by using $dx = cdt$

$$T^{i0} = c\left(\frac{d\vec{p}}{cdtdydz}\right) = \frac{dE}{cdtdA_\perp} = \frac{dE/dA_\perp}{cdt} = \frac{dp/dA_\perp}{dt}$$

So T^{i0} also gives the rate of flow of momentum per unit area crossing a surface.

6.6.3 Energy flux

Energy flow per unit area across a surface is called energy flux.

The component T^{0i} represents the energy flux. To find T^{0i} first put $\mu = 0, \nu = i$ in Eq.(6.1)

$$T^{01} = c\frac{dp^0}{dS^1} = \frac{dE}{cdtdydz} = \frac{dE}{cdt\,(dA_\perp)} = \frac{1}{c}\frac{dE/dA_\perp}{dt}$$

And same for $\nu = 2, 3 = y, z$. Thus T^{0i} is the ith component of rate of flow of energy per unit area across the surface x^i.

6.6.4 Stress

Force flow per unit area is called stress.

The component T^{ij} represents the energy flux. To find T^{ij} first put $\mu = i, \nu = j$ in Eq.(6.1)

$$T^{ij} = c\frac{dp^i}{dS^j} \tag{6.2}$$

Particularly for $i = j =$ (diagonal terms) .

$$T^{ii} = c\frac{dp^i}{dS^i} = c\frac{d\vec{p}}{cdtdA_\perp} = \frac{d\vec{p}/dA_\perp}{dt} = \frac{d\vec{F}}{dt}$$

So T^{ii} is called pressure.

For $i \neq j$ (off-diagonal terms) the stress will be given by Eq.(6.2) and it represents the rate of flow of ith components of momentum per unit area at right angles to the jth direction. It is actually shear stress for $i < j$ and momentum flux for $i > j$.

All of the components of stress-energy tensor are summarized in a matrix below:

6.6.5 Conservation Laws

Conservation laws can be derived using stress-energy tensor using the condition.

$$\partial_\mu T^{\mu\nu} = 0$$

This equation describes that the covariant divergence of stress-energy tensor is zero. This equation implies that energy and momentum are conserved. So

$$\begin{aligned}\frac{\partial T^{\mu\nu}}{\partial x^\mu} &= 0\\ \frac{\partial T^{00}}{\partial x^0} + \frac{\partial T^{0i}}{\partial x^i} &= \frac{\partial T^{00}}{\partial t} + \nabla \cdot T^{0i}\end{aligned}$$

Indices	Components of Stress-Energy Tensor
$\mu = 0, \nu = 0$	$T^{00} = nmv^0 v^0 = nm\gamma c^2$
$\mu = 0, \nu = 1$	$T^{01} = nmv^0 v^1 = nm\gamma c v_x = T^{10}$
$\mu = 0, \nu = 2$	$T^{02} = nmv^0 v^2 = nm\gamma c v_y = T^{20}$
$\mu = 0, \nu = 3$	$T^{03} = nmv^0 v^3 = nm\gamma c (0) = 0 = T^{30}$
$\mu = i, \nu = i$	$T^{ii} = nmv^i v^i = nm\left(v^i\right)^2 = \begin{cases} mnv_x^2, mnv_y^2; i \neq z \\ 0; i = z \end{cases}$
$\mu = i, \nu = j$	$T^{ij} = nmv^i v^j = \begin{cases} mnv_x v_y; i, j \neq z \\ 0; i = z \end{cases} = T^{ji}$

Table 6.1: *Components of Stress-Energy Tensor*

$$\frac{\partial T^{\mu\nu}}{\partial x^{\mu}} = \frac{\partial E}{\partial t} + \nabla \cdot \pi$$

6.6.6 Dust

The perfect pressureless fluid having all the particles at rest and no gravity is called dust.

The particles of such a fluid interact only through gravitational force that's why they are called dust. Their pressure is zero but they have energy and momentum. Their energy and momentum is in simple form that give rise to gravitational force.

The quantities concerned with dust particles are their energy density (ρ) which gives energy of dust particles and velocity (v) which gives momentum of dust particles.

As $M = Nm$ where N is number of dust particles and m is energy (actually mass and energy are equivalent) of single particle then

$$\rho = \frac{Nm}{V} \equiv nm$$

Where n is number of particles per unit volume.

Now to find momentum, the four velocity v^{μ} and v^{ν} are used. So the stress-energy tensor becomes

$$T^{\mu\nu} = \rho v^{\mu} v^{\nu} = nmv^{\mu} v^{\nu}$$

Using $v^{\mu} = (\gamma c, v_x, v_y, 0)$, components of stress-energy tensor can be determined (Table.6.1) Putting all above values in stress-energy tensor

$$T^{\mu\nu} = \begin{pmatrix} nm\gamma c^2 & nm\gamma c v_x & nm\gamma c v_y & 0 \\ nm\gamma c^2 & nm\gamma v_x^2 & nm\gamma v_x v_y & 0 \\ nm\gamma c v_x & nm\gamma v_x^2 & nm\gamma v_y^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For dust particles are at rest. i.e., $\nu = (v_x, v_y, v_z) = (0,0,0)$. So

$$[T^{\mu\nu}] = \begin{pmatrix} nm\gamma c^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

6.6.7 Ideal Fluid

A collection of all particles that interact through gravitational force having mass density ρ moving with four velocity v^μ and pressure p equally distributed is called ideal fluid.

The stress-energy tensor of an ideal fluid is given by

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) v^\mu v^\nu - p g^{\mu\nu}$$

Where $g^{\mu\nu}$ is metric tensor. In Minkowski space $g^{\mu\nu}$ is replaced with Minkowski metric $\eta^{\mu\nu}$ so that

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) v^\mu v^\nu - p \eta^{\mu\nu}$$

Using different values of μ and ν, the stress-energy tensor can be written in a matrix form as

$$[T^{\mu\nu}] = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

Since $T^{00} = \rho c^2 \neq 0$ this is due to flow of energy in fluid in the form of heat and thermal motion. Similarly $T^{0i} = T^{i0} = 0$ and $T^{ij} = 0$ for $i \neq j$ is due to the lack of interactions between particles. $T^{ii} = p$ represents the momentum transfer per unit time per unit area.

6.6.8 Electromagnetic Field Tensor

Stress-energy tensor can also be written for electromagnetic field using electromagnetic field tensor as

$$T^{\mu\nu} = \frac{1}{\mu_o}\left(F^{\mu\alpha} F^{\nu}_{\alpha} - \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right)$$

Where $F^{\mu\nu}$ is the electromagnetic field tensor and $g^{\mu\nu}$ is metric tensor.

In matrix form $T^{\mu\nu}$ is given by

$$T^{\mu\nu} = \begin{pmatrix} \frac{1}{2}\left(\varepsilon_o E^2 + \frac{B^2}{\mu_o}\right) & S_x/c & S_y/c & S_z/c \\ S_x/c & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ S_y/c & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ S_z/c & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{pmatrix}$$

Where S_i is the Poynting vector that represents the direction of electromagnetic field given by

$$S_i = \frac{1}{\mu_o}(E \times B)_i$$

And σ_{ij} is Maxwell stress tensor given by

$$\sigma_{ij} = \varepsilon_o E_i B_j + \frac{1}{\mu_o} B_i B_j - \frac{1}{2}\left(\varepsilon_o E^2 + \frac{B^2}{\mu_o}\right)\delta_{ij}$$

6.7 Einstein's Field Equations

Q. Derive Einstein's field equations in vacuum and matter in the presence and absence of gravity.

A set of field equations which describe the fundamental gravitational interaction in spacetime curve due to energy and momentum are called Einstein's field equation.

In Newtonian gravity, gravitational interaction (force) is between two matter objects. So matter tell matter how to move while in General Relativity, energy and momentum decides gravitational interaction. So momentum and energy decides the motion of matter and radiation.

Einstein's equations describe how the spacetime structure is curved in different regions due to energy and momentum. In spacetime geometry an important idea is geodesic which has been discussed in geodesic equations.

Einstein's field equations are categorized into three types:

- Field equations in the absence of gravitational field in vacuum
- Field equations in the presence of gravitational field in vacuum
- Field equations in the presence of gravitational field in matter.

6.7.1 Field Equations in Vacuum

Newton's law of motion for a free particle in the absence of gravitational field is

$$\frac{d^2 x_i}{dt^2} = 0 \tag{6.3}$$

In general relativity geodesic equation plays the same role as Newton's law of motoin. So equation of motion of a free particle is

$$\frac{d^2x^\mu}{d\alpha^2}+\Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\alpha}\frac{dx^\sigma}{d\alpha}=0 \tag{6.4}$$

This equation is **field equation in vacuum in the absence of gravitational field.**

Eq.(6.4) can reduce to Eq.(6.3) if vanish. i.e.,

$$\Gamma^\mu_{\nu\sigma}=0$$

For above condition Riemann tensor $R^\rho_{\mu\nu\lambda}$ is given by

$$R^\rho_{\mu\nu\lambda}=\partial_\nu\Gamma^\rho_{\mu\lambda}-\partial_\lambda\Gamma^\rho_{\mu\nu}+\Gamma^\sigma_{\mu\lambda}\Gamma^\rho_{\sigma\nu}-\Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\lambda}$$

also vanish. i.e.,

$$R^\rho_{\mu\nu\lambda}=0$$

In Newtonian gravitation theory

$$\nabla^2\phi=0$$

Required

$$\phi=0$$

In the same way in general relativity

$$R^\rho_{\mu\nu\lambda}=0$$

Required

$$R^\lambda_{\mu\nu\lambda}=R_{\mu\nu}=0$$

It is a second order differential equation just like Laplace equation $\left(\nabla^2\phi=0\right)$. Actually it is a second rank Ricci tensor. Above equation is the **field equation in vacuum in the presence of gravitational field.**

6.7.2 Field Equations in Matter

In Newtonian gravitational theory, gravitational potential in the presence of matter is given by Poisson's equation

$$\nabla^2\phi(x)=4\pi G\rho$$

Einstein thought of including second order derivative of metric tensor $(g^{\mu\nu})$, generalization of ϕ on left hand side and stress-energy tensor $(T_{\mu\nu})$; generalization of ρ on right hand side. He made a relation between metric tensor (Ricci tensor $R_{\mu\nu}$; consisting metric tensor) and matter (stress-energy tensor; $(T_{\mu\nu})$) as

$$R_{\mu\nu}=\kappa T_{\mu\nu}$$

Where.κ is some constant.

If energy conservation is checked in above equation it should be

$$\nabla^\mu T_{\mu\nu}=0$$

or

$$\nabla^{\mu} R_{\mu\nu} = 0$$

But in actual

$$\nabla^{\mu} R_{\mu\nu} = \frac{1}{2}\nabla_{\nu} R \neq 0$$

The covariant derivative of Ricci tensor does not vanish. This violates the conservation of energy in suggested Einstein field equations. He, then, made a modification in above field equations to deal with the problem of conservation.

There exist is tensor whose covariant derivative vanish. i.e., Einstein tensor given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{6.5}$$

In the same sense as discussed above for the relation between Ricci tensor ($R_{\mu\nu}$) and stress-energy tensor ($T_{\mu\nu}$) a relation between Einstein tensor ($G_{\mu\nu}$) and stress energy tensor ($T_{\mu\nu}$) will be

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{6.6}$$

Tracing above equation to check whether above equation actually produces gravity or not

$$\begin{aligned} g^{\mu\nu} G_{\mu\nu} &= g^{\mu\nu} \left(\kappa T_{\mu\nu}\right) \\ g^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right) &= \kappa g^{\mu\nu} T_{\mu\nu} \\ R - \frac{1}{2} R\,(4) &= \kappa T \qquad\qquad (g^{\mu\nu} g_{\mu\nu} = 4) \\ R &= -\kappa T \end{aligned}$$

Putting above value and Eq.(6.6) in Eq.(6.5)

$$\begin{aligned} \kappa T_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left(-\kappa T\right) \\ R_{\mu\nu} &= \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) \end{aligned} \tag{6.7}$$

This is called **field equation for matter**.

Determination of Constant

The constant κ can be predicted in weak field ($h_{\mu\nu} \ll 1$). In such a field T^{00} is the rest energy. So focusing on $\mu = \nu = 0$

$$g^{\mu\nu}(x) = \eta^{\mu\nu} - h^{\mu\nu}$$

where $h_{\mu\nu}$ is small correction field and $\eta_{\mu\nu}$ are Minkowski matrix elements. For $\mu = \nu = 0$ it becomes

$$\begin{aligned} g^{00} &= \eta^{00} - h^{00} \\ g^{00} &= -1 - h^{00} \end{aligned}$$

and trace (T) of stress-energy tensor for $h^{\mu\nu} \ll 1$ is

$$T = g^{\mu\nu}T_{\mu\nu} \cong g^{00}T_{00} \cong (-1)T_{00} = -T_{00}$$

Putting in Eq.(6.7) for $\mu = \nu = 0$

$$\begin{aligned} R_{00} &= \kappa\left(T_{00} - \frac{1}{2}g^{00}T\right) \\ &= \kappa\left[T_{00} - \frac{1}{2}(1)(-T_{00})\right] \\ R_{00} &= \frac{1}{2}\kappa T_{00} \\ R_{00} &= R^{\lambda}_{0\lambda 0} = \frac{1}{2}\kappa T_{00} \end{aligned} \tag{6.8}$$

Since $R^{\lambda}_{0\lambda 0}$ can be determined from Riemann curvature tensor as

$$\begin{aligned} R^{\rho}_{\mu\nu\lambda} &= \partial_\nu \Gamma^{\rho}_{\mu\lambda} - \partial_\lambda \Gamma^{\rho}_{\mu\nu} + \Gamma^{\sigma}_{\mu\lambda}\Gamma^{\rho}_{\sigma\nu} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\sigma\lambda} \\ R^{i}_{0j0} &= \partial_j \Gamma^{i}_{00} - \partial_0 \Gamma^{i}_{j0} + \Gamma^{i}_{j\lambda}\Gamma^{\lambda}_{00} - \Gamma^{i}_{0\lambda}\Gamma^{\lambda}_{j0} \end{aligned}$$

As second term $(\partial_0 = \partial_t)$ is time derivative of static field which vanishes and third and fourth terms are quadratic in Γ which can be neglected. Thus

$$R^{i}_{0j0} = \partial_j \Gamma^{i}_{00}$$

So Eq.(6.8) becomes

$$\begin{aligned} R_{00} &= R^{\lambda}_{0\lambda 0} = \partial_\lambda \Gamma^{\lambda}_{00} \\ &= \partial_\lambda \left[\frac{1}{2}g^{\lambda\rho}\left(\partial_0 g_{\rho 0} + \partial_0 g_{0\rho} - \partial_\rho g_{00}\right)\right] \\ &= -\frac{1}{2}\partial_\lambda \left(g^{\lambda\rho}\partial_\rho g_{00}\right) = -\frac{1}{2}\partial_\lambda \partial^\lambda g_{00} \\ &= -\frac{1}{2}\nabla^2(-1 + h_{00}) \\ R_{00} &= -\frac{1}{2}\nabla^2 h_{00} \end{aligned}$$

Putting in Eq.(6.8)

$$\begin{aligned} -\frac{1}{2}\nabla^2 h_{00} &= \frac{1}{2}\kappa T^{00} \\ \nabla^2 h_{00} &= -\kappa T^{00} \end{aligned}$$

Since $h_{00} = \frac{2}{c^2}\phi$. So

$$\begin{aligned} \frac{2}{c^2}\nabla^2\phi &= -\kappa T^{00} \\ \nabla^2\phi &= -\frac{c^2\kappa}{2}\left(c^2\rho\right) \end{aligned}$$

Comparing with Poisson's equation for gravitation $\left(\nabla^2\phi = 4\pi G\rho\right)$

$$\begin{aligned} -\frac{c^2\kappa}{2}\left(c^2\rho\right) &= 4\pi G\rho \\ \kappa &= -\frac{8\pi G}{c^4} \end{aligned}$$

Thus Einstein field equations of general relativity are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}$$

where $\kappa = -\frac{8\pi G}{c^4}$.

This is sometimes called Einstein's laws of gravitation. It is a set of ten second order differential equations. In electromagnetism, Maxwell's equations lack the corresponding equations of motion but in General relativity, Einstein field equations include equations of motion.

KEY TERMS

- **Curved spacetime:** The spatial geometry which is not flat is called curved spacetime.
- **Inertial mass:** The mass that determines the acceleration of a body under the action of a given force is called inertial mass.
- **Gravitational mass:** The mass that determines the acceleration of a body under the action of the gravitational force is called gravitational mass.
- **Weak form of equivalence principle:** All bodies experience the same acceleration when falling freely in the same gravitational field
- **Strong form of equivalence principle:** The law of physics are the same in an accelerated frame and in a uniform and static gravitational field.
- **Stress-energy tensor:** A second rank tensor that describes the density and flux of energy and momentum in spacetime is called stress-energy tensor.
- **Covariant Differentiation:** A way of specifying a derivative along tangent vector of a manifold is called covariant differentiation.
- **Energy density:** Energy per unit volume is called energy density.
- **Momentum density:** Momentum per unit volume is called momentum density.
- **Energy flux:** Energy flow per unit area across a surface is called energy flux.
- **Stress:** Force flow per unit area is called stress.
- **Dust:** The perfect pressureless fluid having all the particles at rest and no gravity is called dust.
- **Ideal fluid:** A collection of all particles that interact through gravitational force having mass density ρ moving with four velocity v^μ and pressure p equally distributed is called ideal fluid.
- **Einstein's field equations:** A set of field equations which describe the fundamental gravitational interaction in spacetime curve due to energy and momentum are called Einstein's field equation.

KEY FORMULAE

Newton's gravitational law

$$F = G\frac{m_1 m_2}{r^2}$$

Inertial mass

$$m = \frac{\mathbf{F}}{\mathbf{a}}$$

Gravitational mass

$$m_g = \frac{\mathbf{F}_g}{\mathbf{g}}$$

Stress-energy tensor

$$T^{\mu\nu} = c\frac{dp^\mu}{dS^\nu}$$

Energy density

$$p^\mu = \int_V T^{\mu\nu}\varepsilon_{\nu\lambda\rho\sigma}dx^\lambda dx^\rho dx^\sigma$$

Momentum density

$$T^{i0} = c\left(\frac{d\vec{p}}{cdtdydz}\right) = \frac{dE}{cdtdA_\perp} = \frac{dE/dA_\perp}{cdt} = \frac{dp/dA_\perp}{dt}$$

Energy flux

$$T^{01} = c\frac{dp^0}{dS^1} = \frac{dE}{cdtdydz} = \frac{dE}{cdt\,(dA_\perp)} = \frac{1}{c}\frac{dE/dA_\perp}{dt}$$

Stress

$$T^{ii} = c\frac{dp^i}{dS^i} = c\frac{d\vec{p}}{cdtdA_\perp} = \frac{d\vec{p}/dA_\perp}{dt} = \frac{d\vec{F}}{dt}$$

Conservation law

$$\frac{\partial T^{\mu\nu}}{\partial x^\mu} = \frac{\partial E}{\partial t} + \nabla\cdot\pi$$

Einstein's field equation in vacuum (in the absence of gravitation)

$$\frac{d^2x^\mu}{d\alpha^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\alpha}\frac{dx^\sigma}{d\alpha} = 0$$

Einstein's field equation in vacuum (in the presence of gravitation)

$$R^\lambda_{\mu\nu\lambda} = R_{\mu\nu} = 0$$

Einstein's field equation in matter

$$R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

$$\frac{d^2x^\mu}{d\alpha^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\alpha}\frac{dx^\lambda}{d\alpha} = 0$$

EXERCISE

Problems

1. Show that the contraction of the Bianchi identity implies $\nabla_\mu G^\mu_\nu = 0$
2. Compute the kinetic energy of a muon (mass $m_\mu = 1.88 \times 10^{-28} kg$) travelling with speed $9c/10$.
3. The proton has mass $m_p = 1.67 \times 10^{-27} kg$. Compute the total (relativistic) energy of a proton moving with speed $v = 3c/5$.
4. At what speed is the total energy of a particle twice the mass energy?
5. In a nuclear fission of uranium−235 caused by the absorption of a neutron, nuclei of krypton and barium are produced, and three neutrons are emitted. The difference in the total mass of the particles present at the start of the process and those present at the end is $3.08 \times 10^{-28} kg$. What is the energy released in this process, in both joules and electronvolts ($1eV = 1.60 \times 10^{-19} J$)?
6. Given that the binding energy of a hydrogen atom is 13.6 eV, what is the difference between the mass of a hydrogen atom and the masses of its constituent electron and proton?
7. In frame of reference S, an electron moving along the x-axis has energy $3m_ec^2$ and momentum magnitude $\sqrt{8}m_ec$. Use the transformations of energy and momentum to find the energy and momentum magnitude observed in frame S' moving with speed $4c/5$ relative to S in the positive x-direction.
8. Prove the equivalence of active and passive gravitational masses to inertial mass of body

Part III

Cosmology

Chapter 7

Introduction to Cosmology

Learning Objectives

In this chapter you will learn

- Newtonian concepts about cosmology
- About Hubble's law
- Cosmological redshift
- The big bang theory and its subsequent events
- Cosmic microwave background
- Stars, Neutron stars, Pulsars
- Black holes
- Quasars
- Dark energy and Dark matter
- WIMPs and MACHOs
- Flat, open and closed universe
- Singularity

KEY TERMS

curved spacetime, inertial mass, gravitational mass, stress-energy tensor, energy density, momentum density, energy flux, stress, dust, ideal fluid,

7.1 Newtonian Cosmology

Q. What is Newtonian cosmology?

Newton, in his book Philosophiae Naturalis Principia Mathematicia, presented the first gravitational theory. He explained Kepler's empirical laws. His concepts about gravitation were appeared as successful theory when Edmund Halley predicted the future comet.

Newton's theory of gravitation describes most of planetary and satellite mechanics. But Newton's cosmological ideas were developed during a correspondence with Richard Bentley. He asked to Newton how his theory of gravity applied to universe as a whole. Newton described that universe is infinite and static. Stars were considered to be suns evenly distributed homogeneously throughout the universe. The fixed stars cancel out their mutual pulls by opposite directions.

However Newton was aware of the problem in his reasoning about cosmology. If there is a region of excess matter anywhere in the universe then that region will become denser and it will attract more and more matter. Thus a uniform distribution of stars would be unstable due to gravity. Newton supported his idea by saying "this frame of things could not always maintain without a divine power to conserve it".

7.2 Hubble's Law

Q. State and explain Hubble's law.

The recession velocity is directly proportional to the proper distance to a galaxy.

The recession velocity is given by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}e_r = \frac{\dot{r}}{r}\mathbf{r}$$

As $\mathbf{r} = a\mathbf{x}$ with comoving position $\mathbf{x}$ is a constant then velocity becomes

$$\mathbf{v} = \frac{\dot{a}x}{ax}\mathbf{r} = \frac{\dot{a}}{a}\mathbf{r} = H\mathbf{r}$$

Where H is the proportionality constant called Hubble's parameter given by

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

It indicates the rate of expansion of the universe. For today (present time) $t = 0$ and Hubble's parameter becomes Hubble's constant ($H_o = \dot{a}(t_o)$). Its value is positive due to expanding universe. The scale factor $a(t)$ describes the time evolution of the universe. It can serve as cosmic clock. It also gives the measure of the size of the universe.

From above equation the units of Hubble's constant are sec^{-1}. However for convenience often used units are $kms^{-1}Mpc^{-1}$. Its value is

$$H_o = 100hkms^{-1}Mpc^{-1}$$

7.3 Cosmological Redshift

Q. What is cosmological redshift?

The phenomenon in which electromagnetic radiations coming from an object undergo an increase in wavelength due to expansion of space is called cosmological redshift.

In Doppler Effect redshift is the phenomenon occur due to receding relative motion of the source and observer. But in general relativity, cosmological redshift is attributed to expansion of the space.

The important notion for the measurement of separation between celestial objects is redshift in the spectrum of light emitted by any distant object. Due to expansion of universe the redshift becomes larger and larger.

The spectral shift is given by

$$z = \frac{\lambda_{rec} - \lambda_{em}}{\lambda_{em}} = \frac{\lambda_{rec}}{\lambda_{em}} - 1 \tag{7.1}$$

When light pass between two objects separated by distance dr then their relative velocity dv, according to Hubble's law, will be

$$v = Hr$$

or

$$dv = H dr$$

$$dv = \frac{\dot{a}}{a} dr \left(\because H = \frac{\dot{a}}{a} \right) \tag{7.2}$$

For nearby points, Doppler Effect implies that

$$\frac{d\lambda}{\lambda_{em}} = \frac{dv}{c}$$

Using Eq.(7.2)

$$\frac{d\lambda}{\lambda_{em}} = \frac{\dot{a}}{a} \frac{dr}{c}$$

The time between emission and reception is given by the light travel time $dt = \frac{dr}{c}$. So above equation becomes

$$\frac{d\lambda}{\lambda_{em}} = \frac{\dot{a}}{a} dt = \frac{1}{a} \frac{da}{dt} dt = \frac{da}{a}$$

Integrating,

$$\ln \lambda = \ln a + const.$$

i.e.,

$$\lambda \propto a$$

As expansion of the universe is described by the scale factor. Above relation implies that as space expands wavelengths become longer. Thus the wavelength scales as $a(t)$

$$\frac{\lambda_{rec}}{\lambda_{em}} = \frac{a(t_{rec})}{a(t_{em})}$$

Putting in Eq.(7.1)

$$z = \frac{a(t_{rec})}{a(t_{em})} - 1$$

For present epoch, scale factor $a(t_{rec}) = 1$. So redshift becomes

$$z = \frac{1}{a(t_{em})} - 1$$

For $z = 1$, above equation gives

$$1 = \frac{1}{a(t_{em})} - 1$$

or

$$a(t_{em}) = \frac{1}{2}$$

Thus universe was half as large as at present. The photon decoupling, after Big Bang, was occurred at redshift $z_\gamma = 1100$.

7.4 The Big Bang Theory

Q. What is the big bang theory? Describe its subsequent processes.

The cosmological model of observable universe characterized by zero time and a subsequent expansion phase is called Big Bang theory.

Big Bang theory is the event happened 13.8 billion years ago and in the result this observable universe came into being. After Big Bang there were many subsequent events happened which are briefly described below:

1. Planck Epoch ($E \gtrsim 10^{19} GeV$): The earliest period of time ($t < 10^{-43} s$) is believed to be extraordinary short time. At Planck epoch quantum effects of gravity dominated physical interactions. After this time gravitational force separated itself from other forces.

2. Grand Unification Epoch ($E > 10^{16} GeV$): The three fundamental forces of standard model are unified called grand unification theory. After this time ($t < 10^{-36} s$) strong nuclear force separated itself from other forces.

3. Inflation ($E = 10^{15} - 10^{9} GeV$): In less than a nano second ($t < 10^{-32} s$) a phase transition caused a cosmic inflation in which the universe expand exponentially. The inflation ended at ($t < 10^{-32} s$). The universe is supercooled from 10^{27} to 10^{22} Kelvins. The strong interaction became distant from the electroweak interaction.

4. Baryogenesis ($E = 200MeV$): After inflation, reheating of universe ($t = 10^{-12} \sim 10^{-6}s$) happened which obtained the temperature necessary for production of a quark-gluon plasma and other elementary particles. But the temperature was so hight that an unknown reaction occurred which violated the conservation of baryon number leading to a small excess of quarks and leptons. This resulted in the predominance of matter over antimatter in the present universe.

5. Neutron-Proton Transition ($E > 1MeV$): At about $t = 10^{-6}s$, quarks combined to form protons and neutrons, the building blocks of atomic nuclei.

6. Neutrino Decoupling ($E\ 1MeV$): At $t = 1s$ ending weak interactions decoupled the neutrinos becoming free-streaming to form the cosmin neutrino background.

7. Nucleosynthesis ($E = 100\ 1keV$): During $t = 0.01 - 200s$, the universe had been continuously cooling and the lightest nuclein of hydrogen (deuterium) and helium formed.

8. Radiation-Matter Equality ($E \simeq 1eV$): For $t = 70k$years the radiation energy density fell below that of non-relativistic matter changing the dynamics of the universe.

9. Photon-Decoupling ($E \simeq 0.3eV$): At $t = 380k$years electrons begin orbiting nuclei, creating atoms, the glow from our universe is unveiled. Photons (glow) could free-stream to form the cosmic microwave background observed today.

10. The Dark Ages ($E = 0.3eV$): For 300 million years the cosmic background radiation was the only light. Clumps of matter that will became galaxies glow brightest.

11. Reionization ($E = 5.17 \times 10^{-3} - 1.64 \times 10^{-3}eV$): After 300 million years dense gas clouds collapse under their own gravity and that of dark matter to eventually form galaxies and stars. Nuclear fusion lights up the stars.

12. Galaxy Formation ($E = 1.64 \times 10^{-3} - 3.45 \times 10^{-4}eV$): At $t = 100$million years formation of stars and galaxies began. At years again the cosmic expansion of the universe started but at an accelerated rate. The universe continued to expand becoming ever less dense. As a result, fewer new stars and galaxies are forming to present time (13.8 billion years).

13. Present Time ($E = 2.33 \times 10^{-4}eV$): After $t = 13.8$billion years passing through all above phases present time is obtained and it is considered as the age of the universe.

An important parameter in the particle interactions is the temperature. It ranges from a few Kelvin to 10^7 Kelvin for matter. However there is a temperature that does not refer to matter but radiation observable today as background radiation. Temperatures for different subsequent events of Big Bang are given in a table below(7.1):

7.5 Cosmic Microwave Background

When the universe expanded and cooled, ions and electrons freezed into neutral atoms and cosmic soup lost its ability to catch photons. These free photons are called cosmic microwave background.

Sr.♯	Epoch	Time	Radiation temperature (Energy)
1	Planck epoch	$< 10-43s$	$> 10^{32}K(> 10^{19}GeV)$
2	Grand unification epoch	$< 10^{-36}s$	$> 10^{29}K(> 10^{16}GeV)$
3	Inflationary epoch, Electroweak epoch	$< 10^{-32}s$	$10^{28}K \backsim 10^{22}K(10^{15} \backsim 10^{9}GeV)$
4	Baryogenesis	$10^{-12}s \backsim 10^{-6}s$	$> 10^{12}K(> 100MeV)$
5	Neutron-Proton Transition	$10^{-6}s \backsim 1s$	$> 10^{10}K(> 1MeV)$
6	Neutrino decoupling	$1s$	$10^{10}K(1MeV)$
7	Big Bang nucleosynthesis	$10s \backsim 10^{3}s$	$10^{9}K \backsim 10^{7}K(100 \backsim 1keV)$
8	Radiation-Matter equality	70,000 years	$10^{4}K(1eV)$
9	Photon decoupling	$10s \backsim 380,000$ years	$10^{9}K \backsim 4000K(100k \backsim 0.4eV)$
10	Dark ages	380,000 $\backsim$ 150 million years	$4000K \backsim 60K$
11	Reionization	150 million $\backsim$ 1 billion years	$60K \backsim 19K$
12	Galaxy formation and evolution	1 $\backsim$ 10 billion years	$19K \backsim 4K$
13	Present time	13.8 billion years	2.7 K

Table 7.1: *Different subsequent events of Big Bang*

The thermal photons separated from cosmic soup when universe expanded and cooled and ions and electrons freezed into neutral atoms are called cosmic microwave background.

The cosmic microwave background radiation was predicated by Gamow, Alpher and Herman in 1948. They called the cosmic microwave background radiation a direct consequence of the Big Bang with a temperature of about 2.7K. In 1964, Robert Dicke detected the cosmic microwave background.

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$$T_{\gamma,0} = 2.725 \pm 0.002K$$

COBE confirmed the thermal nature of the cosmic radiation. The total energy density ε_{rad} of radiation at temperature T is

$$\varepsilon_{rad} = \rho_{rad}c^2 = \alpha T^4.$$

Where α is the blackbody constant given by

$$\alpha = \frac{\pi^2 k_B^4}{15\hbar^3 c^3} = 7.565 \times 10^{-16} Jm^{-3}K^{-4}$$

For above temperature the energy density becomes

$$\varepsilon_{rad} = \left(7.565 \times 10^{-16}\right)(2725)^4 = 4.17 \times 10^{-14} Jm^{-3}$$

The number density of photons for $T = 2.725K$ is

$$n_\gamma = \frac{2404}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 \simeq 410/cm^3 = 4.10 \times 10^8/m^3$$

Thus in the universe there are, on average, 410 photons in every cubic centimetre. The density of baryons is

$$n_B \simeq 0.23/m^3$$

The baryon-proton number ratio will be

$$\frac{n_B}{n_\gamma} = \frac{0.23}{4.10 \times 10^8} = 6.05 \times 10^{-10}$$

For every baryon (proton or neutron) there are almost two billion photons.

7.6 Historical Background of the Universe

Q. What is historical background of the universe. Write a note on stars, neutron stars, pulsars, black holes and quasars.

The universe came into being approximately 13.8 billion years ago by a great event Big Bang. After Big Bang universe passed through different stages including Planck epoch, Grand unification epoch, inflation, electroweak epoch, quark-baryon transition, neutrino decoupling, nucleosynthesis and stars and galaxy formation.

7.6.1 Stars

Star is an astronomical object consisting of luminous spheroid of plasma held together by gravity.

After Big Bang the universe was in a cosmic dark age that lasted upto 800 million years. After that reionization began with the expansion of the universe and vast clouds of hydrogen and helium began to become dense. Hypothetically, super massive black holes were formed before the existence of the first stars. They started off as 'Quasi stars'. The star collapses in itself and was matter at an enormous rate producing a monstrously large black holes. The earliest star forming cloud began condensing 300 million years ago. One cloud reached a critical mass (Jean mass) it began to collapse into itself. As cloud collapsed it began to glow. When temperature reaches $1700°C$ it becomes protostar and deuterium fusion began. At this stage first star was born. With nucleosynthesis, stars began to create metal elements. The early born stars exploded completely into a powerful supernova. The supernova spread the elements within the universe becoming metal free stars. The stars formed began to rotate about massive star giving rise to a galaxy 500 million years after the Big Bang.

7.6.2 Neutron Stars

A celestial object of very small radius and very high density composed of closely packed neutrons is called neutron star.

Neutron stars are formed by the gravitational collapse of the massive star by a supernova explosion. A start, 10 to 20 times massive as compared to sun, when collapse due to its own mass may become a black hole. But due to insufficient mass to become black hole, subsequent process of supernova give birth to neutron star which is usually observable. Once they are formed they no longer generate heat. They are usually observable as pulsars.

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7.6.3 Pulsars

Highly magnetized and rapidly spinning neutron stars that emit a beam of electromagnetic radiations are called pulsars.

Neutron stars are emitting a strong beam of electromagnetic radiation which rotates like a lighthouse beam so that we see a series of pulses coming from stars at night. These pulsing stars are called pulsars This produces a very precise interval between pulses that ranges from milliseconds to seconds for a pulsar. Pulsars are source of ultra high energy cosmic rays.

A schematic view of pulsar is shown in fig. the sphere in the middle is neutron star.

A pulsar is a neutron star that emits beams of radiation that sweep through Earth's line of sight. Like a black hole, it is an endpoint to stellar evolution. The 'pulses' of high-energy radiation we see from a pulsar are due to a misalignment of the neutron star's rotation axis and its magnetic axis. Pulsars seem to pulse from our perspective because the rotation of the neutron star causes the beam of radiation generated within the magnetic field to sweep in and out of our line of sight with a regular period, somewhat like the beam of light from a lighthouse. The stream of light is, in reality, continuous, but to a distant observer, it seems to wink on and off at regular intervals. Pulsars are the original gamma-ray astronomy point sources. A few years after the discovery of pulsars by radio astronomers, the Crab and Vela pulsars were detected at gamma-ray energies. Pulsars accelerate particles to tremendous energies in their magnetospheres. These particles are ultimately responsible for the gamma-ray emission seen from pulsars.

7.6.4 Black Holes

A region of spacetime having a gravitational field so intense that no matter or radiation can escape from it is called black hole.

In 1939, J. Robert Oppenheimer published an article 'on continued gravitational contraction' in which he predicted the contraction of a star under the influence of its own gravitational field. In the result creating a body with an intense attractive force that not even light could escape from it.

In 1939, J. Robert Oppenheimer published an article 'on continued gravitational contraction' in which he predicted the contraction of a star under the influence of its own gravitational field. In the result creating a body with an intense attractive force that not even light could escape from it. The boundary of black hole is called the event horizon. Anything that crosses the event horizon, it collapses into the black hole's singularity which is infinitely small and infinitely dense point for spacetime and laws of physics are no longer applicable there.

There are several different types of black holes with stellar and supermassive black holes being the most common. Stellar black holes are formed when massive stars die and collapse the 10 to 20 times the mass of our sun and scattered throughout the universe. There could be millions of the stellar black holes in the Milky way alone. Super massive black holes are giant billions of times more massive than our sun. Sagittarius is a black hole in the center of our galaxy. It has a mass of about 4 millions suns and has a diameter about the distance between the earth and the sun.

Black holes are invisible, however they can be detected and observed by their nearby matter including accretion disc. The accretion disk is a disk of particles that form when gases and dust fall towards a black hole and quasars jets.

In 1916, using general relativity, Schwarzchild calculated that any mass can becomes black hole if it were compressed tightly enough. The Schwarzchild radius for a blck hole is

$$R = \frac{2GM}{c^2}.$$

In 1917, astronomers discovered the first black hole. An untold number of black holes are scattered throughout the universe which are constantly wrapping space and time altering entire galaxies.

7.6.5 Quasars

A massive and extremely remote celestial object, emitting exceptionally large amount of energy, and typically having a starlike image in a telescope. It has been suggested that represent a stage in the evolution of some galaxies.

It is an extremely luminous active galactic nucleus in which a supermassive black hole with mass ranging from millions to billions of time the mass of the Sun is surrounded by a gaseous accretion disk.

7.7 Dark Energy

Q. What is dark energy. Also find cosmological constant.

A theoretical form of energy postulated to act in opposition to gravity and to occupy the entire universe and causing to accelerate the expansion of the universe is called dark energy.

In 1998, astronomers found that the entire universe is going apart a an accelerating rate. There must be something that cause the universe to expand. Everything that makes such an expansion of universe is dark energy. It is seen never and we don't know exactly what is it? That's why it is called dark due to its ignorance.

Dark energy makes up 72% of the universe (FIG). In 1998, astronomers notice that universe was not just expanding but it was actually accelerating faster and faster outward. They began to think, maybe space itself is expanding.

In Einstein's view, spacetime is basically a flat mesh unless energy or matter bends it. There maybe a special kind of substance that is invisible and undetectable that pushes the spacetime mesh apart. There maybe some virtual particles are creating and annihilating in spacetime which are not explainable using standard model of particle physics. So dark energy is hard to imagine.

Dark energy is a field and it has same density ($\backsim 7 \times 10^{-30} g/cm^3$) everywhere. If it is expanded it will seems to be at the same position but actually will be expanding. It does not get diluted because it is embedded in space. As space expands, we get more space but we get more dark energy with it looks the same. The bigger the universe gets, the faster it flies apart. The distance between our Milkyway and the neighbouring galaxies will be growing faster than the speed of light. When that happens, even our best telescopes would not be able to see these galaxies.

7.7.1 Cosmological Constant

When Einstein started to study the cosmology, he realized that there is a term to be added in the field equations that was mathematically justified. He called it 'the greatest blunder of my life'.

The additional cosmological term Λ modified the field equations as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

The universe was considered to be static (non-expanding) and he found that non-zero value of Λ gives static solutions of field equations. In Newtonian limit, a positive value of Λ gives repulsive effect that can counterbalance the usual gravitational attraction. Later on it was discovered that universe was not only expanding but at an accelerating rate.

From above equation

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= \kappa T_{\mu\nu} - \Lambda g_{\mu\nu} \\ R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= \kappa \left(T_{\mu\nu} + \frac{\Lambda}{\kappa} g_{\mu\nu} \right) \end{aligned}$$

On the right hand side there is an additional term $\left(\frac{\Lambda}{\kappa}g_{\mu\nu}\right)$ contributing in stress-energy tensor. Let's say this term as new stress-energy tensor $\bar{T}_{\mu\nu}$ then above equation becomes

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa\left(T_{\mu\nu} + \bar{T}_{\mu\nu}\right) \tag{7.3}$$

The new stress-energy tensor $\bar{T}_{\mu\nu}$ for an ideal fluid having density ρ_λ and pressure p_λ is

$$\bar{T}_{\mu\nu} = \left(\rho_\Lambda c^2 + p_\Lambda\right)\frac{u_\mu u_\nu}{c^2} - p_\Lambda g_{\mu\nu} \tag{7.4}$$

where $\rho_\Lambda c^2$ is the density of dark energy and p_λ is the pressure due to dark energy.

For $\bar{T}_{\mu\nu} = \frac{\Lambda}{\kappa}g_{\mu\nu}$ Eq.(7.4) becomes

$$\frac{\Lambda}{\kappa}g_{\mu\nu} = \left(\rho_\Lambda c^2 + p_\Lambda\right)\frac{u_\mu u_\nu}{c^2} - p_\Lambda g_{\mu\nu}$$

Comparing the coefficients of $\frac{u_\mu u_\nu}{c^2}$ and $g_{\mu\nu}$

$$\begin{aligned} \rho_\Lambda c^2 + p_\Lambda &= 0 \qquad \frac{\Lambda}{\kappa} = -p_\Lambda \\ \rho_\Lambda c^2 &= -p_\Lambda \\ \rho_\Lambda = \frac{\Lambda}{\kappa c^2} \\ \frac{\Lambda}{\kappa} = \rho_\Lambda c^2 \end{aligned}$$

Since positive density and negative pressure of dark energy means that things are going apart rather than coming together. Putting this value in Eq.(7.4)

$$\begin{aligned} \bar{T}_{\mu\nu} &= (0)\frac{u_\mu u_\nu}{c^2} + \left(\frac{\Lambda}{\kappa}\right)g_{\mu\nu} \\ \bar{T}_{\mu\nu} &= \rho_\Lambda c^2 g_{\mu\nu} \end{aligned}$$

Putting this value in Eq.(7.3)

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa\left(T_{\mu\nu} + \rho_\Lambda c^2 g_{\mu\nu}\right)$$

where $\kappa = -\frac{8\pi G}{c^4}$.

This is **modified field equation with cosmological constant** for general relativity.

7.8 Dark Matter

Q. What is dark matter? Describe in details.

A hypothetical form of matter that makes approximately 85% of the universe and about a quarter of its total energy density is called dark matter.

About 95% of what makes up our universe cannot be accounted for the 5% of matter we can perceive abides by the laws of physics. When laws of gravity are applied to find how fast nearly galaxies are rotating and going apart, the equations of general relativity required a missing mathematical term for a missing matter because the forces should be ripping the galaxies apart but the reason of the mysterious type of matter is unknown. This matter is called dark matter because it has invisible force which bends the galaxies together.

The dark matter even does not interact with electromagnetic radiation. That's why it is very difficult to detect using available astronomical equipment.

Other evidence for the existence of dark matter is gravitational lensing. The model to study dark matter is standard Lambda CMD model of cosmology. According to this model the total mass-energy of the universe contains 5% ordinary matter and energy, 27% dark matter and 68% dark energy. Thus dark matter constitutes 85% of total mass while dark energy plus dark matter constitutes 95% of the total mass-energy contents. The majority of dark matter is non-baryonic composed of dark matter is non-baryonic composed of undiscovered subatomic particles.

The undiscovered particles are weakly interacting massive particles called WIMPS. Many experiments to detect dark matter are being done but none is successful yet.

7.9 WIMPs

Q. What are WIMPs?

Hypothetical weakly interacting massive particles that constitute dark matter are called WIMPs.

A WIMP must have been produced thermally in the early universe similar to particles of the standard model after Big Bang. These particles cannot be predicted with the standard model of particle physics but the supersymmetry (SUSY) predicts a new particle with the properties similar to WIMP. However, Large Hadron Collider (LHC) experiment for its direct detection has cast doubt on the WIMP hypothesis.

WIMPs interact only through weak nuclear force and gravity or some unknown interactions with cross section in the range of weak scale. They have large mass as compared to standard model particles. The lightest WIMP have mass in GeV. They do not interact with electromagnetic radiations so they are invisible through electromagnetic observations. They are massive so they must be slow moving and cold particles.

WIMPs are produced by falling out thermal equilibrium with the hot dense plasma of the early universe.

7.10 MACHOs

Q. What are MACHOs?

Mass astrophysical compact halo objects that might explain the apparent presence of dark matter in galaxy halos are called MACHOs.

A MACHO is a body composed of normal baryonic matter that emits little or no radiations and drifts through interstellar space unassociated with any planetary system. MACHOs are invisible like WIMPs so they are hard also to detect through electromagnetic observations. Black holes, neutron stars, brown dwarfs and unassociated planets all are MACHOs.

A MACHO maybe detected when it passes near the star and MACHO's gravity bends the light causing the star to appear brighter due to gravitational lensing.

7.11 Concept of Flat, Open and Closed Universe

Q. What is flat, open and closed universe?

The shape of the universe is decided by the geometry of spacetime which depends on its curvature due to mass and energy. The local geometry of the universe is described by the curvature of spacetime and that of global geometry by topology. The size of the universe considering center at earth is 46 billion light years ($4.4 \times 10^{26} m$) but the age of the universe is only 13.8 billion years. This means that space itself has expanded.

Globally the universe may have following properties:

- Finite or infinite
- Flat, open or closed
- Connected space

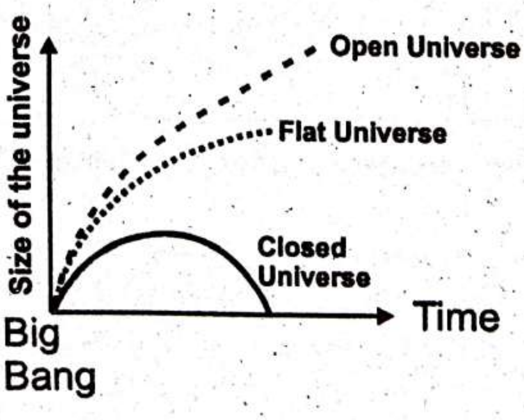

Figure 7.1: *Flat, Open and closed Universe*

The coordinates used to explain 3-manifold model in 4-dimensional spacetime are comoving coordinates. A model used to explain the shape of the universe is Friedmann-Lemaitre-Roberterson-Walker (FLRW) model.

The curvature is a quantity that describes how much flat the shape is. The space has three cases:

- Zero curvature; flat with sum of angles in a triangle equal to 180°
- Positive curvature; 3D space like sphere with sum of angles in a triangle
- Negative curvature; 3d space like hyperbola with sum of angles in a triangle less than 180°.

In general relativity curvature of the universe is determined by the mass-energy density parameter (Ω).

Density parameter is the ratio of average density of the universe and critical energy density.

There are three possible values of density parameter:

- If $\Omega = 1$, the universe is flat
- If $\Omega > 1$, there is a positive curvature and the universe is closed
- If $\Omega < 1$, there is a negative curvature and the universe is open.

The experimental data from WMAP and Planck spacecraft gives normal mass (baryonic matter and dark matter), relativistic particles (photons and neutrinos) and dark energy (cosmological constant) as

$$\Omega_{mass} \approx 0.315 \pm 0.018, \quad \Omega_{rel} \approx 9.24 \times 10^{-5}, \quad \Omega_\Lambda \approx 0.6817 \pm 0.0018$$

Then

$$\begin{aligned} \Omega_{total} &= \Omega_{mass} + \Omega_{rel} + \Omega_\Lambda \\ \Omega_{total} &= 1.00 \pm 0.02 \end{aligned}$$

From above value of density parameter universe is predicted to be flat with only a 0.4% margin of error.

7.12 Singularity

The infinitely small point in spacetime where gravity, mass density and spacetime curvature become infinity is called singularity.

It is also called gravitational singularity. It is independent of the coordinate system means that it cannot be removed by any change of coordinates. Laws of physics also cease to work in singularity. It is unsure that singularity exists at present time or existed at the time of Big Bang. Singularity would form at the center of the black hole when an object (like star) collapse in spacetime.

According to Friedmann-Robertson-Walker (FRW) model, the universe started expanding form the condition in which scale factor was zero (at some time in past). This is also called the initial singularity. The time is known as the Big Bang.

KEY TERMS

- **Hubble's law:** The recession velocity is directly proportional to the proper distance to a galaxy.
- **Cosmological redshift:** The phenomenon in which electromagnetic radiations coming from an object undergo an increase in wavelength due to expansion of space is called cosmological redshift.
- **Big bang theory:** The cosmological model of observable universe characterized by zero time and a subsequent expansion phase is called Big Bang theory.
- **Cosmic microwave background:** When the universe expanded and cooled, ions and electrons freezed into neutral atoms and cosmic soup lost its ability to catch photons. These free photons are called cosmic microwave background.
- **Stars:** Star is an astronomical object consisting of luminous spheroid of plasma held together by gravity.
- **Neutron stars:** A celestial object of very small radius and very high density composed of closely packed neutrons is called neutron star.
- **Pulsars:** Highly magnetized and rapidly spinning neutron stars that emit a beam of electromagnetic radiations are called pulsars.
- **Black hole:** A region of spacetime having a gravitational field so intense that no matter or radiation can escape from it is called black hole.
- **Quasar:** A massive and extremely remote celestial object, emitting exceptionally large amount of energy, and typically having a starlike image in a telescope. It has been suggested that represent a stage in the evolution of some galaxies.
- **Dark energy:** A theoretical form of energy postulated to act in opposition to gravity and to occupy the entire universe and causing to accelerate the expansion of the universe is called dark energy.
- **Dark matter:** A hypothetical form of matter that makes approximately 85of its total energy density is called dark matter.
- **WIMPs:** Hypothetical weakly interacting massive particles that constitute dark matter are called WIMPs.
- **MACHOs:** Mass astrophysical compact halo objects that might explain the apparent presence of dark matter in galaxy halos are called MACHOs.
- **Density parameter:** Density parameter is the ratio of average density of the universe and critical energy density.
- **Flat universe:** The universe is flat if density parameter $\Omega = 1$.
- **Open universe:** The universe is flat if density parameter $\Omega < 1$.

- **Close universe:** The universe is flat if density parameter $\Omega > 1$.
- **Singularity:** The infinitely small point in spacetime where gravity, mass density and spacetime curvature become infinity is called singularity.

KEY FORMULAE

Hubble's law

$$H(t) = \frac{\dot{a}(t)}{a(t)}$$

Spectral (cosmological) redshift

$$z = \frac{\lambda_{rec} - \lambda_{em}}{\lambda_{em}}$$

Baryon-proton number ratio

$$\frac{n_B}{n_\gamma} = 6.05 \times 10^{-10}$$

Schwarzchild radius

$$R = \frac{2GM}{c^2}$$

Cosmological constant

$$\Lambda = \kappa \rho_\Lambda c^2$$

Density parameter

$$\Omega_{total} = 1.00 \pm 0.02$$

Short Question From Whole Book

1. How do astronomers tell the difference between the different kinds of redshifts?
2. Could an astronaut use a wrist watch to measure relativistic time dilation?
3. What is it about quantum mechanics that is incompatible with general relativity?
4. How did Roemer measure the speed of light back in the 17th century?
5. Does the star DI Herculis prove that general relativity is breaking down?
6. What property of matter causes geodesics to curve near them?
7. Exactly how does light get red shifted in a gravitational field?
8. Can space exist by itself without matter or energy around?
9. How did Bradley determine the speed of light in 1779 from stellar parallax?
10. What objects have we seen traveling near the speed of light either natural or man-made?
11. Do gravitational fields produce their own gravity?
12. What happens in the space between gravitating bodies?
13. Is space-time orientable?
14. Is the gravity of distant objects the cause of inertia?
15. If space exists, what is it?
16. We used to think that the speed of sound was impassable, isn't the speed of light the same thing?
17. What is 'sub-space'?
18. How can matter just appear out of nowhere?
19. What are P-branes?
20. Do any theories require more than 1 dimension of time?
21. What does light travel time have to do with looking into the past or future?
22. Do time and space exist for a photon on which eternity should pass as an instant?
23. If space ends, what's on the other side?
24. Why does the warping of space near the Earth make its orbit elliptical, but has no effect upon light?
25. Are all the postulates in General Relativity accepted or does there exist room for alternatives to General Relativity with different outcomes?
26. If astronomers can use the cosmic background radiation as a reference frame doesn't that invalidate special relativity?
27. If there is so little friction in space, why can't we reach the speed of light with a big rocket?
28. Why do the laws of physics break down in singularities?
29. What proofs are there for general relativity?
30. What is the relationship between space and time?
31. If I could put myself in the same point in spacetime where I was yesterday at noon, would it still be yesterday at noon?
32. If you travel fast in space will you REALLY come back younger than the friends you left behind?
33. Can gravitational lens events cause gamma ray bursts to reoccur in the same part of the sky, rather than repeated explosions of the same object?
34. If gravity travels at the speed of light, why don't the equations used in celestial mechanics include time delays?
35. Is space really quantized?
36. Does space have more than 3 dimensions?

37. When your speed exceeds light, how does your mass become 'imaginary'?
38. As you approach the speed of light does the universe seem to disappear into a bright point of light ahead of you?
39. Why can't we travel faster than light?
40. If the universe is finite, can gravity waves have already traveled completely around the universe and back?
41. If space can be bent, can it also be sliced?
42. What does the word 'spacetime' mean?
43. Does faster than light travel mean time travel is possible?
44. If I shined a flashlight into the sky, would someone in another galaxy eventually see it?
45. If I passed the Sun at near the speed of light, it's mass would be higher, so why wouldn't the planets fall in?
46. Are there any other astronomical systems where general relativity can be tested other than the Hulse-Taylor binary pulsar?
47. What is the simplest evidence that there are more than 4 dimensions?
48. If Observer A was accelerating relative to Observer B, would they agree that it is Observer A that gained mass?
49. What is it like to be in a place with no gravity?
50. Why isn't our 4-dimensional space-time curved in the 5th dimension?
51. Are there really such things as gravitons?
52. How do you explain Einstein's relativity theory to a 12 year old?
53. If a marshmallow traveled at 99.99 percent the speed of light hit the Earth, what would happen?
54. Why don't gravitational lenses also produce redshifts?
55. Can gravity waves be used to carry information?
56. What prevents densities higher than the so-called Planck Limit from happening?
57. If gravity is a distortion in space-time, why do we call it a force?
58. According to relativity, how much younger are astronauts in earth-orbit after 6 months?
59. How do you really know that the limits to general relativity for strong fields are set inside black holes?
60. Does light travel at the same speed back and forth along a straight line?
61. Because of general relativity, do stars age at the same rate inside and at the surface?
62. What is meant by the 'Euclidean geometry of space'?
63. What would happen if the speed of light were 10 times faster?
64. If an object travels near light speed, what does its image look like?
65. Is there any common sense way to think about the Twin Paradox?
66. Does time stop when you travel at the speed of light?
67. Could you tell me the name of the NASA astronomer who discovered two gaps in time?
68. Is there a theory that predicts the velocity of light?
69. If we traveled 100 light years into space, would we really see the Earth as it was 100 years ago?
70. Why is the Doppler shift the only explanation for cosmological redshifts?
71. Does gravity produce a Doppler shift?
72. Why is the gravitational field of the universe another name for space-time?
73. How are scientists trying to detect gravity waves?
74. What is curved space-time?

75. Don't objects steadily evaporate over time by emitting gravity waves?
76. Why do photons produce pressure even though they are massless?
77. Has it been experimentally proven that the speed of light is the fastest?
78. Why do Singularities have to have zero volume?
79. Can loops in time exist in nature?
80. Is there a static condition against which the speed of light is measured?
81. How do you reconcile the universe not having an expansion center with observations that galaxies are moving away from us?
82. Does anyone really understand general relativity well enough to have an intuitive grasp of the universe?
83. If nothing can travel faster than light, why does gravity take affect instantly?
84. If two ships approach at 2/3 the speed of light, would they see each other?
85. In layman's terms, why does your rate of speed affect time?
86. What kind of experiment would show that time is quantized?
87. Why dose the decay of an orbit mean that gravitational radiation is being produced?
88. Will we ever be able to move through time?
89. How soon after you start to move do you see the relativistic contraction effect?
90. If the speed of light is a constant, why is it slower in glass and water?
91. Do Newtonian and Einsteinian gravity yield the same results for the same input parameters?
92. What binary star is producing gravity waves?
93. Is there any way we can measure our true speed through the universe?
94. Can light be altered by a magnetic field?
95. Why can't a rigid body exist?
96. Why does time have to change at high speeds?
97. What does the equation look like that shows how gravitational radiation is lost from the binary pulsar system?
98. When an object falls under gravity, is it really feeling a force, or is it just moving along a curved path through space?
99. Under what circumstances can a particle exceed the speed of light?
100. What would a universe be like with two time dimensions?
101. What happens to rotating solid objects in relativity theory?
102. How is gravitational lensing used to search for dark matter?
103. Why does the 'Twin Paradox' exist as a physical phenomenon?
104. Has the constancy of the speed of light been measured?
105. Do radio waves get red shifted?
106. Exactly why is the speed of light a constant in all reference frames?
107. If you could slow everything down to a complete stop, would time pass infinitely fast to you?
108. What is the fourth dimension?
109. How is it said that relativity is tested billions of times a year, when there are only a few accelerator labs on Earth?
110. Why can't time be considered a force?
111. How does general relativity incorporate rotation?
112. Can you use the geometry of gravitational lenses to determine the distances to the lensed quasar?
113. If spacetime is a consequence of the deformation of space by gravity, why is gravity a universal

phenomenon rather than a local one?
114. What is the momentum of a photon?
115. How could you get Doppler shifts with light if the speed of light never changes?
116. How do we know gravitational lenses aren't just refraction off of intergalactic clouds?
117. In the problem of the marshmallow moving at the speed of light, didn't you forget about its kinetic energy?
118. Can you exceed the speed of light by manipulating space-time in some way?
119. Do time and space become meaningless at the speed of light?
120. Why is time so different from space in the way we experience it?
121. How does a body actually bend space-time?
122. Does relativity apply to 'sideways' angular motion or just motion along the line of sight?
123. What the ding dang are gravitons?
124. Why aren't all gravitational lenses perfect rings?
125. Is gravity the exchange of gravitons, or the curvature of space-time?
126. Do photons carry mass as suggested by the equation m = hf/c2 ?
127. When a photon falls in a gravitational well, does its speed exceed 'c'?
128. If gravity is a 'wave' of energy, does the concept of frequency have any meaning?
129. Would the collision of two photons cause them to slow down as a result?
130. Do astronomers take into account gravitational redshifts when they compute the cosmological redshift of a distant galaxy?
131. If we see a galaxy moving near the speed of light, as they would also see ours, whose mass is getting bigger?
132. Why are multiple gravitational lenses more common than rings?
133. If you slowed down light, would it become less of a wave and more of a particle?
134. If an object approached the speed of light, would it be transformed into light?
135. Does gravity travel faster than the speed of light?
136. Does matter really get infinitely compressed in a singularity?
137. How does gravity propagate through space?
138. Does the explanation for superluminal motion in some quasars work for galaxies moving away from us at high speeds due to the expansion of the universe?
139. If there is no such thing as velocity except with respect to a reference object, how can there be speed limits in deep space?
140. What is it that physically prevents faster than light speeds?
141. Are we gravitationally affected by where an object is now, or where it was when the light we see started its journey to us?
142. In Robert Forward's book 'Dragon Egg' the inhabitants of a neutron star see an approaching rocket stand still in space. Does this violate relativity?
143. How are electromagnetic fields and space-time related, and can electromagnetic fields exert a force on space- time?
144. How do you reconcile classical quantum mechanics with general relativity?
145. Why isn't it possible to travel in time?
146. Is the constancy of the speed of light dictated by special relativity invalidated by the expansion of the universe?

147. Where can I get a copy of Misner, Thorne and Wheeler's 'Gravitation' book published in 1973?
148. If a photon travels at the speed of light, why isn't its mass infinite?
149. What is the speed of light in megaparsecs per planck time?
150. What fraction of the perihelion advance of Mercury is due to special relativistic effects?
151. Why can't you choose the reference frame where time moves fastest as the one at absolute rest?
152. What is the Einstein Shift?
153. What is time?
154. What is Dark Matter and does it produce gravitational lensing?
155. If two objects approach at 75 percent the speed of light, why are they not traveling 1.5 times the speed of light?
156. If light has traveled thousands of years, how could it possibly have aged according to relativity?
157. Is it ever possible to determine whether a body is at rest?
158. Does time pass for all massive bodies at the same rate?
159. If you travel near the speed of light, do you enter another dimension?
160. If you traveled at the speed of light, would the universe appear instantaneous?
161. Do objects really contract when they travel near the speed of light?
162. What happens to the fabric of space-time when an object moves through it near the speed of light?
163. Is it true that the speed of light is slowing down?
164. What visual distortions occur as you approach the speed of light?
165. Is the speed of light a barrier of the same kind as the speed of sound?
166. Without a fixed reference frame for judging speed in the universe, how do we know which 'twin' did the moving in the famous Twin Paradox?
167. If Einstein says that everything is relative, how can there exist any firm constants upon which to base a Unified Theory of Nature?
168. Could the displacement of star images near the Sun be caused by refraction in the atmosphere of the sun, not by general relativity?
169. What can you learn from studying gravity waves?
170. What studies have been made for gravity waves?
171. Why does time change when you travel near the speed of light?
172. Could energy ever come to a point and create mass?
173. What does Stephen Hawking currently think about time travel?
174. Would our system of time change if we went to another solar system?
175. Do worm holes allow all times to exist simultaneously?
176. Does 'now' have a size instead of being a point in spacetime?
177. Do electrons interact with past and future representations of themselves?
178. If a blue shifted clock runs fast, why do fast moving particles decay slowly?
179. Can gravity affect the speed of light?
180. Is there such a thing as hyperspace?
181. How do we know that the speed of light is the same in every galaxy?
182. Is our perception of distances and sizes in the universe just an illusion?
183. What is a space anomaly?
184. What is a vortex, and how does it work?
185. How do distant objects get magnified by space curvature?

186. Why can't matter travel back in time?
187. Is time linear, or could there be other dimensions to it as for space?
188. Is it possible to exceed the speed of light?
189. Can you see Cherenkov radiation in empty space?
190. What is a space time continuum?
191. Do the different estimates for the age of the universe imply that the speed of light may have changed?
192. Is the amount by which light is bent important in cosmology?
193. How is it that, according to general relativity, space can be curved?
194. Is the cosmological redshift real, or just an optical illusion caused by dust?
195. Is it difficult to know what is happening now if it takes light time to get here?
196. Why does a particles mass increase as it moves?
197. How do you actually use Einstein's famous equation $E = mc^2$?
198. Could you explain how gravitational lenses work, and what determines the number of images you see?
199. Is anti-matter the same as negative mass?
200. If we could travel faster than light, could we go back in time?
201. Why is the speed of light a constant?
202. Is there a limit to how fast you can go using gravitational boosts from the sun and planets?
203. Is there a simple way to measure the speed of light?
204. Does gravity travel at the speed of light?
205. What is light, how does it move, and how do we measure its speed?

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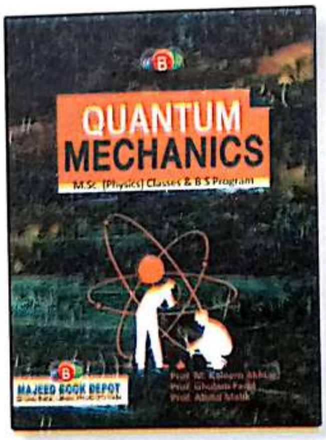
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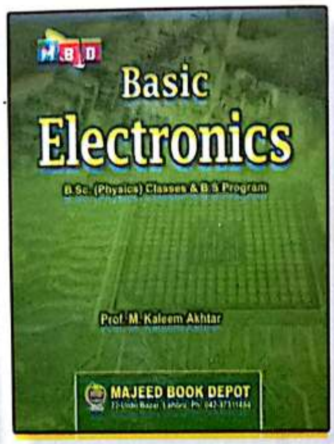
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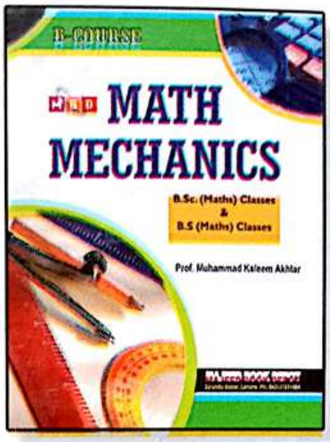
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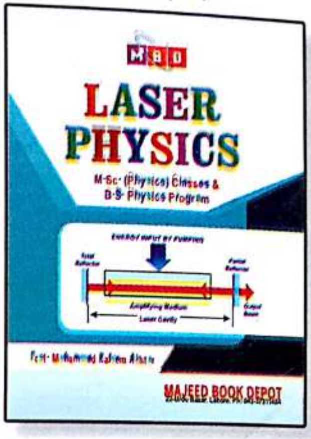
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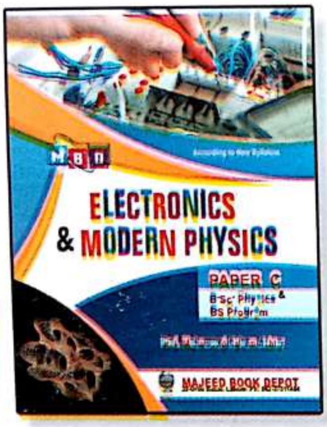
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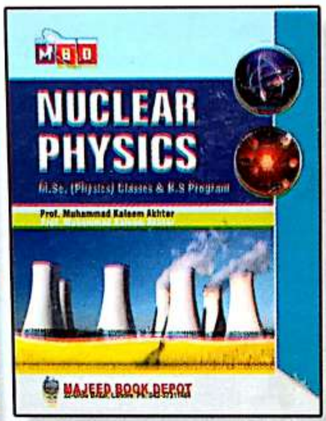
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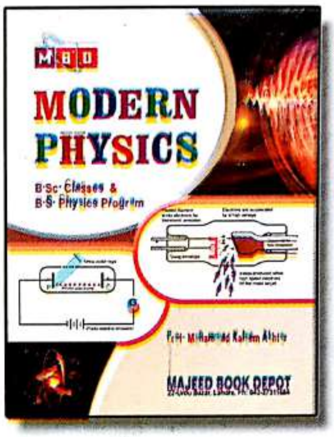
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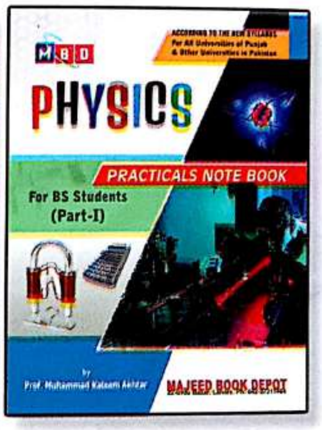
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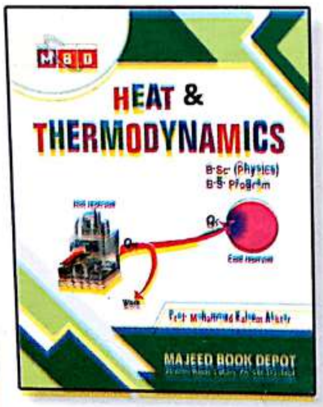
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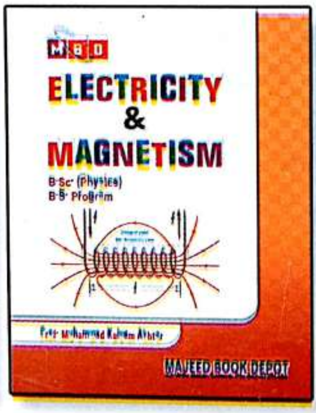
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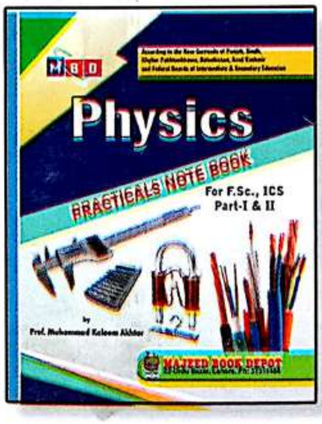
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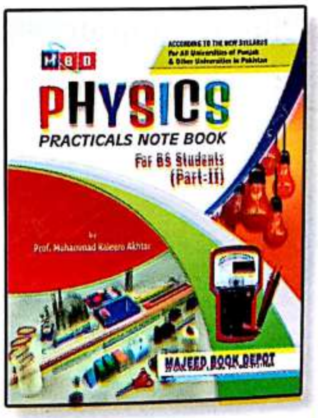
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